Also useful for other **MBA** entrance examinations like **XAT, SNAP** and **IIFT**

PEARSON SUCCESS COMPANION

QUANTITATIVE APTITUDE for the CAT

Nishit K. Sinha

- **5 simulated CAT online tests** in CD
- **4000+** fully solved questions
- Special section for **Campus Recruitment Examinations**
- **7 years'** fully solved CAT papers
- Updated as per **latest pattern**

4e

CAT Simulator in CD

ALWAYS LEARNING

PEARSON

Published by Pearson India Education Services Pvt. Ltd, CIN: U72200TN2005PTC057128, formerly known as TutorVista Global Pvt. Ltd, licensee of Pearson Education in South Asia.

ISBN 978-93-325-7001-6
eISBN 978-93-325-7640-7

Head Office: A-8 (A), 7th Floor, Knowledge Boulevard, Sector 62, Noida 201 309, Uttar Pradesh, India.
Registered Office: Module G4, Ground Floor, Elnet Software City, TS-140, Block 2 & 9, Rajiv Gandhi Salai, Taramani, Chennai 600 113, Tamil Nadu, India.
Fax: 080-30461003, Phone: 080-30461060
www.pearson.co.in, Email: companysecretary.india@pearson.com

To

Papa and Ma

Kumar Kalyan Prasad Sinha and Sanjila Sinha

This page intentionally left blank

Contents

Preface to the Fourth Edition *viii*
Preface *ix*
About the Author *x*
Acknowledgements *xi*
CAT Demystified *xii*
How to Prepare for QA *xvii*
How to Use This Book *xviii*

PART 1
QUANTITATIVE APTITUDE

Module 1 Numbers and their Properties

Chapter 1	Vedic Mathematics	1.3-1.9
Chapter 2	Number System	1.10-1.54
	Benchmarking Test 1	*1.55-1.56*

Module 2 Arithmetic

Chapter 3	Percentage	1.58-1.87
Chapter 4	Simple Interest and Compound Interest	1.88-1.94
Chapter 5	Profit, Loss, and Discount	1.95-1.116
Chapter 6	Average and its Applications	1.117-1.146
Chapter 7	Ratio, Proportion, and Variation	1.147-1.172
Chapter 8	Time and Work	1.173-1.201
Chapter 9	Time, Speed, and Distance	1.202-1.235
	Benchmarking Test 2	*1.236-1.239*

Module 3 X+2 Maths

Chapter 10 Basics of Algebra 1.241-1.248
Chapter 11 Equations 1.249-1.276
Chapter 12 Sequence and Series 1.277-1.302
Chapter 13 Function 1.303-1.332
Chapter 14 Graphs and Maxima Minima 1.333-1.354
Chapter 15 Logarithm 1.355-1.368
Chapter 16 Permutation and Combination 1.369-1.395
Chapter 17 Probability 1.396-1.412
Benchmarking Test 3 *1.413-1.415*

Module 4 Measurement

Chapter 18 Geometry 1.417-1.488
Chapter 19 Mensuration 1.489-1.516
Chapter 20 Co-ordinate Geometry 1.517-1.526
Benchmarking Test 4 *1.527-1.530*

Module 5 Miscellaneous

Chapter 21 Set Theory 1.532-1.550
Chapter 22 Statistics 1.551-1.557
Chapter 23 Miscellaneous 1.558-1.570

PART 2
CAT PAPERS

CAT 2002 *2.3-2.12*
CAT 2003 *2.13-2.35*
CAT 2004 *2.36-2.44*
CAT 2005 *2.45-2.52*
CAT 2006 *2.53-2.58*
CAT 2007 *2.59-2.66*
CAT 2008 *2.67-2.72*

PART 3
OTHER MBA ENTRANCE PAPERS

Model SNAP Paper 1 *3.3-3.9*
Model SNAP Paper 2 *3.10-3.17*

Model XAT Paper 1 3.18-3.21
Model XAT Paper 2 3.22-3.29
Model IIFT Paper 3.30-3.38

PART 4
SECTION TESTS

Section Test 1 4.3-4.9
Section Test 2 4.10-4.17
Section Test 3 4.18-4.24

PART 5

Special Section for Campus Recruitment Examinations 5.3-5.27

Preface to the Fourth Edition

Common Admission Test (CAT) is known to usher surprises by bringing about changes in its format. CAT 2015 was another such test, which reverted to the 3-section format exam with sectional time-limit from the erstwhile 2-section exam. CAT 2015 also introduced, for the first time, questions in Quantitative Aptitude and Verbal Ability that are not option based. Although the focus areas in CAT 2015 proportionately remained the same as that in the previous years, the test's new format has been set forth with significant changes such as clubbing Data Interpretation with Logical Reasoning once again. Hence, we felt the need to bring out this new edition of *Quantitative Aptitude for the CAT*.

This new edition has incorporated major modifications that will make it more comprehensive and beneficial for all users of the book. The nature and extent of these modifications can be encapsulated as follows:

(a) Chapters have been re-organized with emphasis on QA section in keeping with the current pattern of CAT.
(b) A new chapter on Simple Interest and Compound Interest has been added.
(c) Complete solutions have been added to chapters on Permutation and Combination, and Probability.
(d) A special section catering to Campus Recruitment Tests has been appended at the end.
(e) Model test papers (based upon previous years' examinations) for XAT, IIFT, and SNAP have been included for the benefit of students who intend to appear for exams of other B-schools.
(f) A CD designed to simulate the same GUI experience as that of CAT 2015 has been included with the book. The CD contains five Section Tests based on the pattern of CAT 2015 (both options based and non-options based) to provide students with a hands-on experience and enhance their confidence.

I am sure that the changes made to this new edition will help students to derive more from this book. In case you have any question, you can connect with me on Facebook or Quora.

For any feedback or clarification, I can be reached at nsinha.alexander@gmail.com

Happy Learning!

Preface

The fact that there was no surprise element in CAT 2010 with respect to CAT 2009 was itself a surprise. Probably IIMs/ CAT committee wanted to regain the ground that they might have lost during CAT 2009 online test glitches.

So everybody was expecting some changes in CAT pattern in 2011, and CAT committee obliged. The CAT changed again in 2011—with the introduction of (a) 2-section format instead of 3-section format and (b) sectional time limit. This is a welcome change as now the CAT is similar to other global exams like the GMAT and GRE. They also comprise only two sections and have sectional time limit.

This book covers everything that you need to prepare for the Quantitative Aptitude and Data Interpretation Section as per CAT 2011 pattern. Before we move ahead, let us see the cut-off percentile of different IIMs for CAT 2011 (for general category):

S. No	IIM's	Overall score	QA/DI	VA/LR
1	IIM-A	99	94	94
2	IIM-B	90	80	90
3	IIM-C	99.55	94.24	93.73
4	IIM-L	90	85	85
5	IIM-I	90	85	85
6	IIM Trichy	80	70	70
7	IIM Kashipur	75	70	70
8	IIM Udaipur	80	70	70
9	IIM Ranchi	90	70	70
10	IIM Shillong	***	65	65
11	IIM Kozhikode	85	55	55

*** Not given.

This book is divided into four parts: Part 1—Quantitative Aptitude; Part 2—CAT Papers; Part 3—Other MBA Entrance Papers; and Part 4—Section Tests. First part is further distributed among modules to facilitate the learning of students. Once a student is through with chapters, there are benchmarking tests with calibrated percentile. Moving on further, there are CAT papers to know the level of your preparedness. Finally there are three section tests to help you assess your preparation level.

Further, to facilitate the online testing, a CD containing Section Tests and Full Length Tests is appended.

Your comments and suggestions would be very useful in improving subsequent editions of this book. Please mail me your suggestions at: nsinha.alexander@gmail.com

Nishit K. Sinha

About the Author

Nishit K. Sinha, an IIM Lucknow alumnus, has been training students for the CAT and other B-school entrance examinations for more than a decade. During this period, he has successfully trained more than 10,000 students of varying backgrounds to clear various MBA entrance examinations. To best analyse the pattern of all the major B-school entrance tests, as well as to remain updated on their pattern, he sits for examinations such as the CAT and XAT every year.

Currently, he is associated with Graphic Era Hill University, Dehradun.

Acknowledgements

This book bears the imprint of many people—my colleagues, my students, and my teachers who have had a significant impact on my thought process and have generously extended help whenever I needed.

Special mention of thanks to Prof. (Dr) Kamal Ghanshala and Prof. (Dr) Sanjay Jasola for providing motivation and guidance to keep myself updated and bring out this new edition.

I would like to thank my teachers Mr Anoop Singhania, Mr Vinay Singh, Mr M.K. Alam Bhutto, Mr Jairam Singh, and Mr Arun Sharma.

My special thanks to my brothers Ravi Shankar Prasad, Sharat Chandra Mayank, Amit Kumar, and Vinit Kumar.

I extend my heartfelt thanks to my colleagues Mr P.A. Anand, Mr Narendra Bisht, Ms Divya Paul, Ms Nishu Chawla, Ms Aanchal, and Mr Anurag Chauhan. This book would not have been possible without the contribution from all my students, past and present, who have helped me improve the content and the presentation of the book, and its new edition.

I would like to thank Ms Sharel Simon and Ms G. Sharmilee for giving the book the final shape. Thanks to Mr Vikas Sharma and Mr H. Nagaraja for ensuring that I get the timely and accurate feedback of the users.

Thanks would be a small word for my wife, who took care of family and home, giving me enough time to complete this project. Love to my son who gave me company during late nights. Sudhir, my man-Friday, who took care of my small necessities, your contribution is noteworthy.

I may have forgotten some names here. I wish to express my gratitude to all who have contributed in the making of this book.

NISHIT K. SINHA

CAT Demystified

CAT stands for the Common Admission Test. It is a test conducted by IIMs for admission into several programs offered by them. Besides IIMs, there are a good number of colleges which accept CAT score in their first round of selection process. As of now, there are 19 IIMs offering PGP at the following places: Ahmedabad, Bangalore, Calcutta, Lucknow, Indore, Kozhikode, Shillong, Ranchi, Rohtak, Raipur, Udaipur, Trichy, Kashipur, Bodhgaya, Nagpur, Sambalpur, Sirmaur, Amritsar, and Vishakhapatnam.

History of the CAT

Almost for the past three decades, since the CAT has been started, it has changed its colours many a time in terms of number of questions, sections asked, and orientation of those questions. Here, we will discuss the pattern of CAT 2000 onwards.

Chart 1

	Number of sections	Total number of questions	Total marks	Time allowed
CAT 2000	3	165	N.A.	120 minutes
CAT 2001	3	165	N.A.	120 minutes
CAT 2002	3	150	N.A.	120 minutes
CAT 2003	3	150	N.A.	120 minutes
CAT 2004	3	123	150	120 minutes
CAT 2005	3	90	150	120 minutes
CAT 2006	3	75	300	150 minutes
CAT 2007	3	75	300	150 minutes
CAT 2008	3	90	360	150 minutes
CAT 2009	3	60	450 (scaled score)	135 minutes
CAT 2010	3	60	450 (scaled score)	135 minutes
CAT 2011	2	60	450 (scaled score)	140 minutes
CAT 2012	2	60	450 (scaled score)	140 minutes
CAT 2013	2	60	450 (scaled score)	140 minutes
CAT 2014	2	100	300	170 minutes
CAT 2015	3	100	300	180 minutes

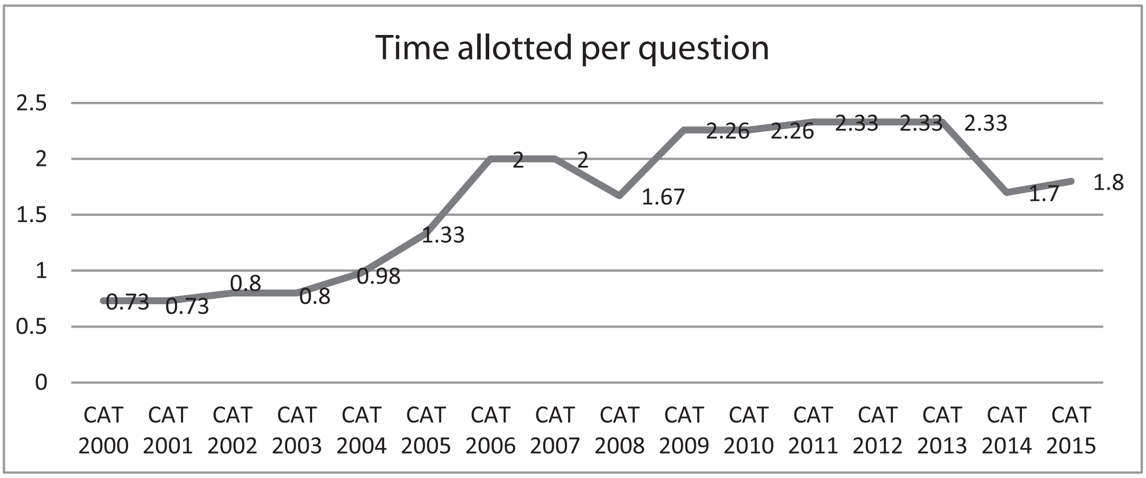
The CAT online examinations

The CAT examinations held from 2011 to 2014 had two sections: (a) Quantitative Ability and Data Interpretation (b) Verbal Ability and Logical Reasoning with 30 questions in each section. It also have sectional time limit of 70 minutes for each section. CAT 2015 had 100 questions: 34 questions in QA, 34 questions in VA/RC, and 32 in LR/DI. Each section had a sectional time limit of one hour.

Before CAT 2004, the CAT did not mention how many marks one question carried? Marks carried per question were announced for the first time in CAT 2004.

Quite obvious from the above table that time allotted per question has risen sharply from CAT 2000 to CAT 2011. One possible conclusion drawn from here is that the CAT is focussing more on accuracy than speed, and secondly, it expects students to gain a certain level of competence across all the areas in a particular section. With the number of questions going down and time going up, students do not have much choice of questions to choose from.

Chart 2



Sectional Breakups and Getting an IIM Call

One thing that has remained constant during this period of CAT 2000 to CAT 2010 is the number of sections and the way these sections have been joined—Quantitative Aptitude (QA), Logical Reasoning and Data Interpretation (LR/DI), and English Usage/Reading Comprehension (EU/RC). Though CAT 2011 changed it all:

Chart 3

Year	QA	LR/DI	EU/RC	Total number of questions
CAT 2000	55	55	55	165
CAT 2001	50	50	50	150
CAT 2002	50	50	50	150
CAT 2003	50	50	50	150
CAT 2004	35	38	50	123
CAT 2005	30	30	30	90
CAT 2006	25	25	25	75
CAT 2007	25	25	25	75
CAT 2008	25	25	40	90
CAT 2009	20	20	20	60
CAT 2010	20	20	20	60
CAT 2011	30 (QA + DI)		30 (Verbal + LR)	60
CAT 2012	30 (QA + DI)		30 (Verbal + LR)	60

Year	QA	LR/DI	EU/RC	Total number of questions
CAT 2013	30 (QA + DI)		30 (Verbal + LR)	60
CAT 2014	50 (QA + DI)		50 (Verbal + LR)	100
CAT 2015	34	32	34	100

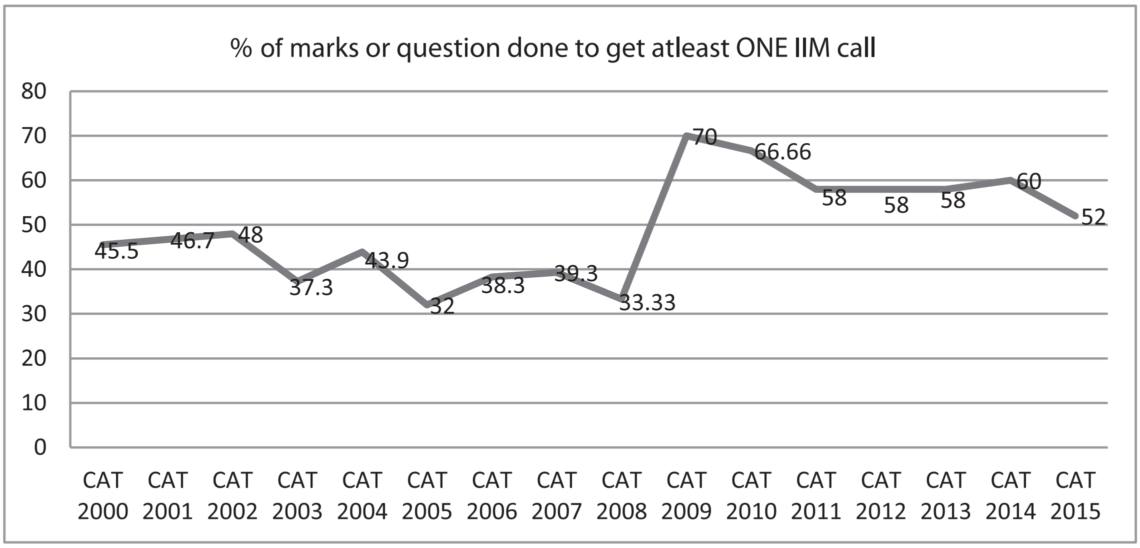
However, past CAT trends show that a student is required to get around 70% marks to get at least one IIM call (with clearing the sectional cut-off). The following table and bar chart give us some clarity regarding the same:

Chart 4

Year	Total marks or questions	Marks/Qs required to get at least one IIM call
CAT 2000	165	75
CAT 2001	150	70
CAT 2002	150	72
CAT 2003	150	56
CAT 2004	123	54
CAT 2005	150	48
CAT 2006	300	115
CAT 2007	300	118
CAT 2008	360	120
CAT 2009	60	42
CAT 2010	60	40
CAT 2011	60	35
CAT 2012	60	35
CAT 2013	60	35
CAT 2014	100	60
CAT 2015	100	52

Following line chart gives questions solved or marks required as a percentage of total marks or total questions (as applicable):

Chart 5



Note: Above calculation is based upon the data collected from the students who got IIM calls in that particular year.

So, to get at least one IIM call in CAT 2008, a student was required to get 33.33% marks out of the total with clearing the cut-off across the sections. Although in the online format of the CAT (since 2009), percentage questions to be done to get at least one IIM call has gone up, it is primarily because the exam is perceived to be easier in its totality than pen and paper-based exams. A student might have got 10 easy questions out of 20 questions in a section.

If we convert the requirement of marks to be obtained from the above line chart into questions to be done, we get the following table:

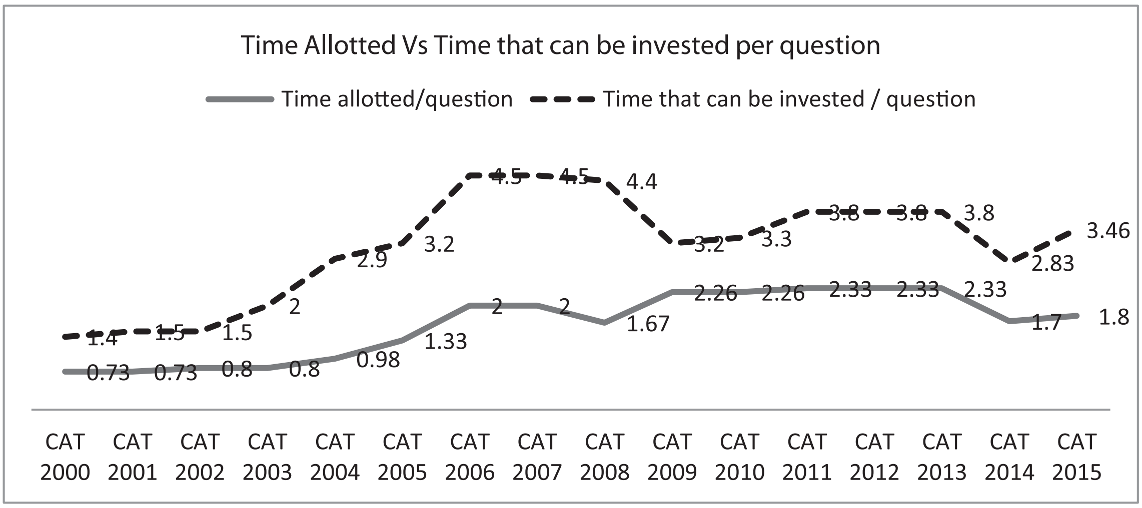
Chart 6

Year	Number of questions to be solved	Time allotted	Time invested per question
CAT 2000	85	120 minutes	1.4
CAT 2001	80	120 minutes	1.5
CAT 2002	81	120 minutes	1.5
CAT 2003	60	120 minutes	2.0
CAT 2004	42	120 minutes	2.9
CAT 2005	38	120 minutes	3.2
CAT 2006	33	150 minutes	4.5
CAT 2007	33	150 minutes	4.5
CAT 2008	34	150 minutes	4.4
CAT 2009	42	135 minutes	3.2
CAT 2010	40	135 minutes	3.3
CAT 2011	35	135 minutes	3.8
CAT 2012	35	135 minutes	3.8
CAT 2013	35	135 minutes	3.8
CAT 2014	60	170 minutes	2.83
CAT 2015	52	180 minutes	3.46

Chart 6—For CAT 2000 to CAT 2008, at 90% accuracy with 1/4 negative marking, these are the approximate number of questions to be done. For CAT 2009 to CAT 2011, net of these many questions is to be done.

To summarize this whole discussion till now, we will compare Chart 2 with Chart 6 and present them in a unified line chart given below (Chart 7).

Chart 7



This line brings to us an important information—for CAT 2006, CAT 2007, or CAT 2008, even if a student has taken approximately 4.5 minutes to solve a question with 90% accuracy, s/he has got enough marks to get at least one IIM call (provided s/he clears the sectional cut-off too). For online formats for CAT, it is around 3.3 minutes per question. For CAT 2015, it was around 3.5 minutes.

So, it's more about accuracy now than speed.

How to Prepare for QA

Purpose of QA Being Included in the CAT

Have you ever thought why Quantitative Aptitude (QA) questions are asked in the CAT? Is it to check your mathematical ability with questions from Algebra, or Permutation and Combination? If that were the case, they would have asked questions also from Group Theory and Applied Mathematics, which they don't. The questions asked in Quantitative Aptitude help to test the student's reasoning ability based on mathematical skills. They also test the student's comprehension of basic mathematical concepts. And even if somebody has not studied mathematics after class 10, they can solve these problems with a little practice. What is required is an understanding of basic concepts. It is, therefore, clear that through the test of QA, the examiners intend to check functional ability pertaining to basic mathematical operations.

Five-point Action Plan for QA

1. Address your math-phobia first You will not be in the best state of mind to crack the CAT if past failures or inabilities in mathematics haunt you. As it is said, it's okay to lose a couple of battles. Keep your eyes set on the war!

2. Make a roadmap 'Divide and Win' is the key. Your preparation should be divided into topics and every topic should be further divided into sub-topics. Once you have a topic-centric roadmap of the entire QA syllabus, carefully distribute the time you want to devote to each section on a daily basis.

3. Strengthen your fundamentals Till you have understood the concepts and their various applications thoroughly, do not start solving the problems. Let this process take some time. Now, before you start solving questions, go over the fundamental concepts once again and see which concept should be applied to get the right answer.

4. Adopt a systematic practice technique Don't jump to solving problems immediately. Make an effort to understand the basic theories behind the mathematical concepts, however trivial they may seem. And only then solve the problems.

5. Focus on weaknesses If students are weak in a particular area (say, geometry or permutations and combinations), they tend to neglect these questions and leave the solutions to choice. This is not only unproductive but also damaging. Identify the areas with which you are not comfortable. Numbers, geometry and algebra account for a major portion of the questions in the CAT; hence, they deserve due attention.

Consistency with Perseverance is the Key

If you do not find ways to solve a problem on the first day, sleep on it and tackle it the next day. Sometimes even the simplest of answers elude you and lead to frustration. Even the best of mathematicians face this problem; therefore, leave it for another day. And if you are unable to solve this problem even on the second day, try it on the third day or the third time. If you are still not able to solve it, check the solution provided at the end of the book or take help. The bottom line is that only consistent efforts and practice can bring positive results. If you want to go through this book in 10 days' time, you can do that also. And you will be able to learn the fundamentals, but remember it will be only short-term. You will be required to go through the same chore again after a few days and repeat this exercise.

How to Use This Book

Let me share with you what my feelings were before I started working on this book. I thought of the kind of book I would prefer if I had to learn the theories of public administration when I do not have any background of this subject. I realized I would want a book which is strong in fundamental concepts but lucid in language; I would need a book which is self-explanatory but not verbose; I would need a book which is all-encompassing but not irrelevant; I would need a book with a good number of practice questions but no repetitive questions (if I cannot learn a concept with 200 relevant questions, chances are low that I will be able to learn that concept with 250 questions having the same level of relevance); I would need a book which can help me judge my progress from time to time. And this is what I have sincerely tried to provide in this book.

How to Go Ahead with This Book

Step 1 Go through the learning objectives and remain focused on them. After completing the chapter, check if these objectives have been fulfilled.

Step 2 Start with the concepts. Before moving on to the next concept, go through the worked-out examples related to that concept. Move ahead only when you have internalized them. Sometimes this might appear to be drudgery, but you must do it.

Step 3 Do the warm-up exercise (in QA part only). It is a precursor to the problems. Most of the questions in the warm-up exercise will help you check only your understanding of the concept and not the application of it. Ideally, you should not give more than 2 minutes to any question in the warm-up exercise. If you get less than 75 per cent questions correct, go over the concepts for which you got the answers wrong.

Step 4 Do the foundation exercise. Solve the problems in this set without any time constraint. This level tests your comprehension of the concept, and is a precursor to the application-based questions. Try to solve all the questions in one sitting, whatever time it takes (one or two breaks of 10 minutes can be taken). If you don't get the correct answers, attempt the same question once again in the next sitting (preferably the next day). This will help you to develop:

- A thorough understanding of the concepts
- An experience of the problems being asked at the basic level
- Confidence building

Step 5 Next is the moderate exercise. This level tests your ability to apply a particular concept, or a combination of concepts, to a problem. You might find that concepts of geometry are being used in problems on time, speed and distance. This will help you to develop:

- The ability to identify easy and difficult questions
- Mental images of the problems
- A logical connection between concepts and their application.

Step 6 Do not go for the advanced level immediately. Relax. Do not solve any QA question for one day. The following day, take the benchmarking test. Your performance in this test can tell you a lot about your progress. Ideally, you should aim for 85 percentile or more.

Step 7 After you have done all the moderate exercises and benchmarking tests, go over the topics again. Only then attempt the advanced questions. Most of the questions here are above the regular CAT level. The idea is to prepare you for a higher level. If you excel at this level, CAT will be a cakewalk for you.

Step 8 Take the Review Tests. Target anything above 98 percentile. If you get it consistently in all the RTs, you are prepared for the CAT. If not, repeat the process from Step 5. If in any one of the RTs, you get less than 85 percentile, repeat the process from Step 4.

All the best!

PART

1

QUANTITATIVE APTITUDE

MODULE 1: Numbers and their Properties

MODULE 2: Arithmetic

MODULE 3: X+2 Maths

MODULE 4: Measurement

MODULE 5: Miscellaneous

MODULE

1

NUMBERS AND THEIR PROPERTIES

- Vedic Mathematics
- Number System

CHAPTER

1

Vedic Mathematics

LEARNING OBJECTIVES

After completion of this chapter, the reader should be able to understand:

- How to do faster calculations?
- Multiplications
- Squares
- Cubes
- Properties of squares and cubes

INTRODUCTION TO THE TOPIC

Vedic mathematics is the ancient system of mathematics drawn from the Vedas. The Vedas are ancient texts that encompass a broad spectrum of knowledge, covering all aspects of life. These include the sutras (verses) pertaining to mathematics. In the early 20th century, Swami Shri Bharati Krishna Tirthaji Maharaja claimed to have rediscovered a collection of 16 ancient mathematical sutras from the Vedas and published it in a book titled *Vedic Mathematics*. Historians, however, do not agree on whether or not these were truly a part of the Vedic tradition. If these sutras dated back to the Vedic era, then they would be a part of an oral rather than a written tradition. In spite of these controversies, they are a novel and useful approach to computation; they are flexible in application and easy to remember. They can often be applied in the algebraic contexts and in simple arithmetic as well.

TYPES OF CALCULATIONS

The different types of calculations that form the basis of mathematics are:

1. Addition
2. Subtraction
3. Multiplication
4. Division
5. Ratio comparison
6. Percentage calculations

When we talk about the techniques of calculations, addition and subtraction can simply not have any short-cuts. Since addition and subtraction are the basic units, we can at best only approximate the values.

In case of multiplication, the techniques of Vedic maths can be used.

Ratio comparison techniques are discussed in the chapter on ratio, proportion, and variation, and the percentage calculations in the chapter on percentage.

VEDIC MATH TECHNIQUES IN MULTIPLICATION

There are several techniques of multiplication. We will discuss them one by one.

Method 1: Base Method

In this method, one number is used as a base; for example, 10, 50, 100, etc. The number that is closer to both the numbers should be taken as the base.

Example 1 105 × 107

Solution In this case, both the numbers are close to 100, so 100 is taken as the base. We will now find the deficit/surplus from the base.

Base = 100, Surplus = 5 and 7

105	+5
107	+7
112	35

The right part (after slash) ⇒ this is the product of the surplus. Since the base = 100 and the surpluses are 5 and 7, the product would be 5 × 7 = 35.

The left part (before slash) ⇒ It could be either of the numbers plus the surplus of the other multiplicand. Hence, the left part would be either (105 + 7) or (107 + 5) = 112 (both will always be the same), i.e., 112.

The left part would be equivalent to the number × 100. In this case, 112 × 100 = 11,200

Now, we add both the right part and the left part = 11,200 + 35 = 11,235

Hence, the result of the multiplication would be 11,235.

Example 2 108 × 104

Solution

108	+8
104	+4
112	32

Example 3 111 × 112

Solution

111	+11
112	+12
123	132

Here, it is 11 × 12 = 132. But it can have only two digits. Thus, 1 will be carried over to the left part and the right part will be only 32. Left part will be either 111 + 12 + 1 (1 for the carry over) or (112 + 11 + 1), i.e., 124. So, the result will be 12,432.

For 102 × 104, the answer will be 10,608. Please note that the right part will be 08 and not simply 8.

Example 4 97 × 95

Solution

97	–3
95	–5
92	15

Base = 100, Deficit = 97 – 100 = –3 and 95 – 100 = –5

Example 5 97 × 102

Solution

97	–3
102	+2
99	–06

97 × 102

Base = 100, Deficit = 97 – 100 = –3,

Surplus = 102 – 100 = 2

The right part will now be (–3) × 2, i.e., –06. To take care of the negative, we will borrow 1 from the left part, which is equivalent to borrowing 100 (because we are borrowing from the hundred digits of the left part). Therefore, this part will be 100 – 06 = 94.

So, the answer = 9894

Example 6 62 × 63

Solution

62	+12
63	+13
75	156

We will assume here the base as 50 owing to the fact that the numbers are close to 50.

Base = 50, Surplus = 62 – 50 = 12,

Surplus = 63 – 50 = 13

The left-hand side = 156 and the right-hand side = 75. Since the base is assumed to be equal to 50, so the value of the right-hand side = 75 × 50 = 3750. Besides, only two digits can be there on the right-hand side, so 1(100) is transferred to the left-hand side leaving 56 only on the left-hand side.

So, the value on the right-hand side = 3750 + 100 = 3850

Value on the left-hand side = 56

Net value = 3850 + 56 = 3906

Let us do the same multiplication by assuming 60 as the base.

62	+2
63	+3
65	06

Base = 60, Surplus = 62 – 60 = 2, Surplus = 63 – 60 = 3

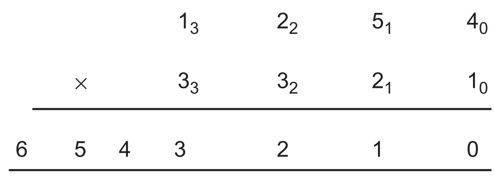
Since the base is assumed to be equal to 60, the value of the right-hand side = 65 × 60 = 130 × 30 = 3900

So, net value = 3906

Method 2: Place Value Method

In this method of multiplication, every digit is assigned a place value and the multiplication is done by equating the place values of multiplicands with the place value of the product.

Let us see this with some examples:

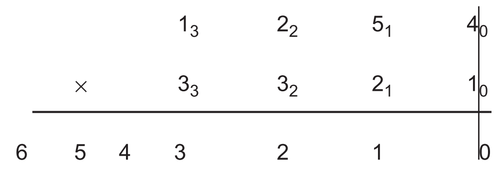


Conventionally, the unit digit is assigned a place value 0, the tens place digit is assigned a place value 1, the hundreds place digit is assigned a place value 2, the thousands place digits is assigned a place value 3 and so on.

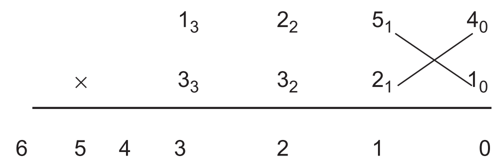
This multiplication is a two-step process.

Step 1 Add the place values of the digits of the numbers given (1254 × 3321) to obtain the place value of the digits of the product.

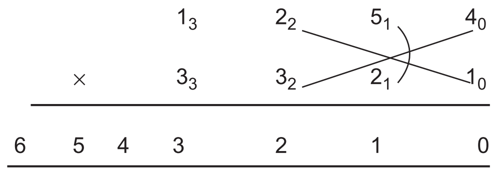
For example, using the place values of the multiplicands, i.e., using 0, 1, 2, and 3 of the number 1254 and the same place values 0, 1, 2, and 3 of the another multiplicand 3321, we can get 0 place value in the product in just one way, i.e., adding 0 and 0.



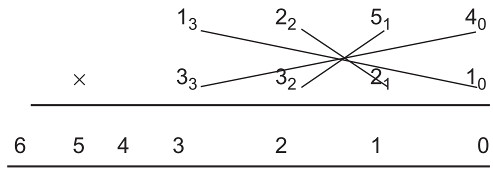
Place value 1 in the product can be obtained in two ways.



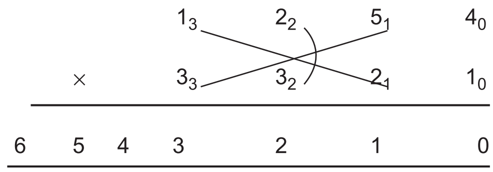
Place value 2 can be obtained in three ways.



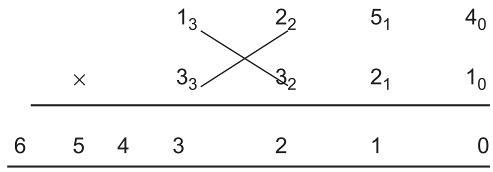
Place value 3 can be obtained in four ways.



Place value 4 can be obtained in three ways.



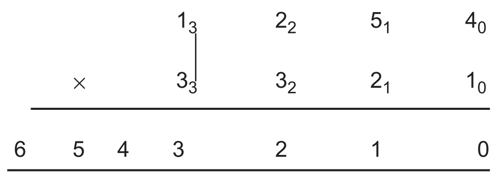
Place value 5 can be obtained in two ways.

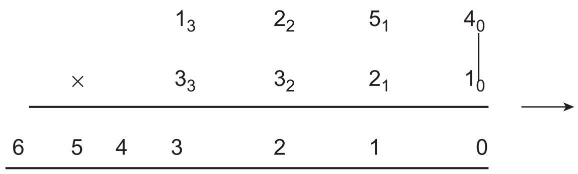


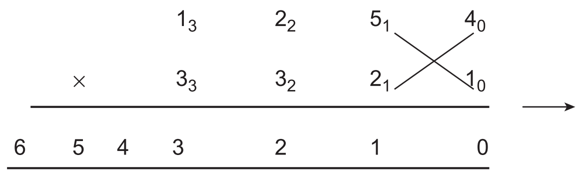
Place value 6 can be obtained in one way.

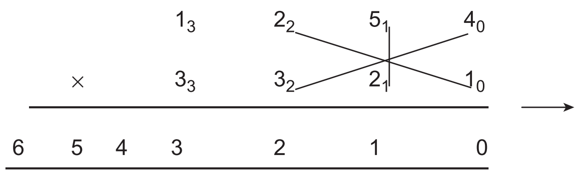
And this is the maximum place value that can be obtained.

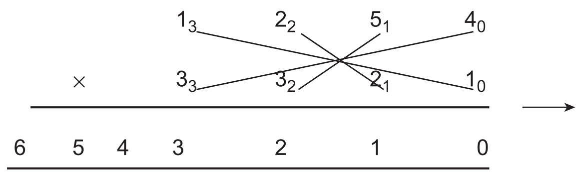
Step 2 Multiply the corresponding numbers one by one.







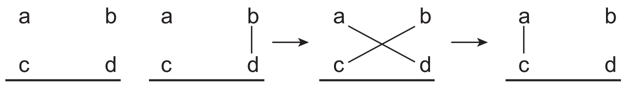




In this manner, we can find the product = 41,64,534

This method is most useful in case of the multiplications of 2 digits × 2 digits or 2 digits × 3 digits or 3 digits × 3 digits multiplication.

Example *ab* × *cd*



Similarly, we can have a proper mechanism of multiplication of 2 digits × 3 digits or 3 digits × 3 digits using the place value method.

Method 3: Units Digit Method

This method of multiplication uses the sum of the unit's digit, provided all the other digits on the left-hand side of the unit digit are the same.

Example 7 75 × 75

Solution

1.0 + 7	5
7	5
56	25

The sum of the units digit = 10, so we add 1.0 in one of the digits on the left-hand side.

Example 8 62 × 63

Solution

0.5 + 6	3
6	2
39	06

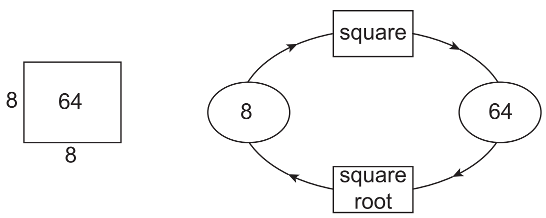
The sum of the units digit = 5, so we add 0.5 in one of the digits on the left-hand side.

SQUARING

A **square number**, also called a **perfect square**, is an integer that can be written as the square of some other integer. In other words, a number whose square root is an integer is known as the square number of a perfect square.

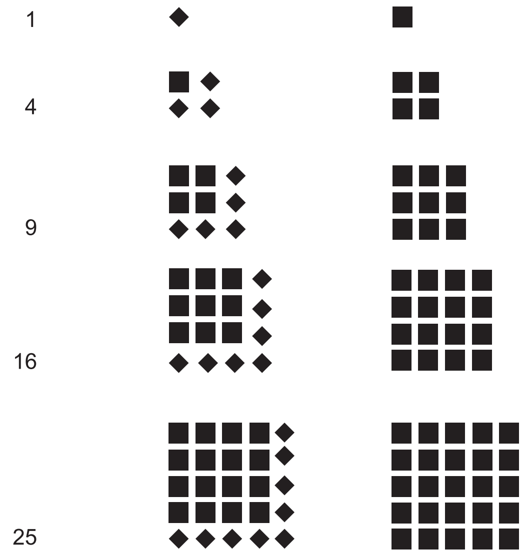
For example, 9 is a square number since it can be written as 3 × 3.

This can be seen through the following flow-chart also.



Properties of a Square Number

1. The number *N* is a square number if it can be arranged as *N* points in a square.



Therefore, it can be deduced that the formula for the *n*th square number is n^2. This is also equal to the sum of the first *n* odd numbers $n^2 = \sum_{k=1}^{n}(2k-1)$ as can be seen in the above figure, where a square results from the previous one by adding an odd number of points (marked as '◆'). For example, $5^2 = 25 = 1 + 3 + 5 + 7 + 9$.

It should be noted that the square of any number can be represented as the sum $1 + 1 + 2 + 2 + \ldots + n - 1 + n - 1 + n$. For instance, the square of 4 or 4^2 is equal to $1 + 1 + 2 + 2 + 3 + 3 + 4 = 16$. This is the result of adding a column and row of thickness 1 to the square graph of three. This can be useful for finding the square of a big number quickly. For instance, the square of $52 = 50^2 + 50 + 51 + 51 + 52 = 2500 + 204 = 2704$.

2. A square number can only end with digits 00, 1, 4, 6, 9, or 25 in base 10, as follows:
3. If the last digit of a number is 0, then its square ends in 00 and the preceding digits must also form a square.
4. If the last digit of a number is 1 or 9, then its square ends in 1 and the number formed by its preceding digits must be divisible by four.
5. If the last digit of a number is 2 or 8, then its square ends in 4 and the preceding digit must be even.
6. If the last digit of a number is 3 or 7, then its square ends in 9 and the number formed by its preceding digits must be divisible by four.
7. If the last digit of a number is 4 or 6, then its square ends in 6 and the preceding digit must be odd.

8. If the last digit of a number is 5, then its square ends in 25 and the preceding digits (other than 25) must be 0, 2, 06, or 56.
9. A square number cannot be a perfect number. (If the sum of all the factors of a number excluding the number itself is equal to the number, then the number is known to be a perfect number.)
10. The digital sum of any perfect square can be only 0, 1, 4, 9, and 7. (Digital sum of any number is obtained by adding the digits of the number until we get a single digit. Digital sum of 385 = 3 + 8 + 5 = 1 + 6 = 7.)

An easy way to find the squares is to find two numbers that have a mean of it. This can be seen through the following example:

To find the square of 21, take 20 and 22, then multiply the two numbers together and add the square of the distance from the mean: $22 \times 20 = 440 + 1^2 = 441$. Here, we have used the following formula $(x - y)(x + y) = x^2 - y^2$ known as the difference of two squares. Thus,

$$(21 - 1)(21 + 1) = 21^2 - 1^2 = 440.$$

Odd and Even Square Numbers

Squares of even numbers are even, as $(2n)^2 = 4n^2$.

Squares of odd numbers are odd, as $(2n + 1)^2 = 4(n^2 + n) + 1$.

Hence, we can infer that the square roots of even square numbers are even and square roots of odd square numbers are odd.

Methods of Squaring

Like multiplication, there are several methods for squaring also. Let us see the methods one by one.

Method 1: Base 10 Method

Understand it by taking few examples:

- Let us find out the square of 9. Since 9 is 1 less than 10, decrease it still further to 8. This is the left side of our answer.
- On the right-hand side, put the square of the deficiency that is 1^2. Hence, the answer is 81.
- Similarly, $8^2 = 64$, $7^2 = 49$
- For numbers above 10, instead of looking at the deficit we look at the surplus. For example,
  $11^2 = (11 + 1); 10 + 1^2 = 121$
  $12^2 = (12 + 2); 10 + 2^2 = 144$
  $14^2 = (14 + 4); 10 + 4^2 = 18; 10 + 16 = 196$
  and so on

This is based on the identities $(a + b)(a - b) = a^2 - b^2$ and $(a + b)^2 = a^2 + 2ab + b^2$.

We can use this method to find the squares of any number, but after a certain stage, this method loses its efficiency.

Method 2: Base 50n Method here, (n is any natural number)

This method is nothing but the application of $(a + b)^2 = a^2 + 2ab + b^2$.

This can be seen in the following examples:

Example 9 Find the square of 62.

Solution Because this number is close to 50, we will assume 50 as the base.

$$(62)^2 = (50 + 12)^2 = (50)^2 + 2 \times 50 \times 12 + (12)^2$$
$$= 2500 + 1200 + 144$$

To make it self-explanatory, a special method of writing is used.

$(62)^2$ = [100's in (Base)]^2 + Surplus | Surplus^2

= 25 + 12 | 144 = 38 | 44 [Number before the bar on its left-hand side is number of hundreds and on its right-hand side is the last two digits of the number.]

$(68)^2$ = 25 + 18 | 324 = 46 | 24

$(76)^2$ = 25 + 26 | 676 = 57 | 76

$(42)^2$ = 25 − 8 | 64 = 17 | 24 [$(a - b)^2 = a^2 - 2ab + b^2$]

Example 10 Find the square of 112.

Solution Since this number is closer to 100, we will take 100 as the base.

$$(112)^2 = (100 + 12)^2 = (100)^2 + 2 \times 100 \times 12 + (12)^2 = 10{,}000 + 2 \times 1200 + 144$$

$(112)^2$ = [100's in (Base)]^2 + 2 × Surplus | Surplus^2

= 100 + 2 × 12 | 12^2 = 125 | 44

Alternatively, we can multiply it directly using base value method.

Had this been 162, we would have multiplied 3 in surplus before adding it into [100's in (Base)]^2 because assumed base here is 150.

$(162)^2$ = [100's in (Base)]^2 + 3 × Surplus | Surplus^2

= 225 + 3 × 12 | 12^2 = 262 | 44

Method 3: 10^n Method

This method is applied when the number is close to 10^n.

With base as 10^n, find the surplus or deficit (×).

Again, the answer can be arrived at in two parts.

$(B + 2x)\ |x^2$

The right-hand part will consist of n digits. Add leading zeros or carry forward the extra to satisfy this condition.

$$108^2 = (100 + 2 \times 8) \mid 8^2 = 116 \mid 64 = 11{,}664$$

$$102^2 = (100 + 2 \times 2) \mid 2^2 = 104 \mid 04 = 10{,}404$$

$$93^2 = (100 - 2 \times 7) \mid (-7)^2 = 86 \mid 49 \Rightarrow 8649$$

$$1006^2 = (1000 + 2 \times 6) \mid 6^2 = 10|12 \mid 036 = 10{,}12{,}036$$

The right-hand part will consist of 2 digits. Add leading zeros or carry forward the extra to satisfy this condition.

$$63^2 = (25 + 13) \mid 13^2 = 38 \mid 169 = 3969$$

$$38^2 = (25 - 12) + (-12)^2 = 13 \mid 144 = 1444$$

Square Mirrors

$$14^2 + 87^2 = 41^2 + 78^2$$

$$15^2 + 75^2 = 51^2 + 57^2$$

$$17^2 + 84^2 = 71^2 + 48^2$$

$$26^2 + 97^2 = 62^2 + 79^2$$

$$27^2 + 96^2 = 72^2 + 69^2$$

Some Special Cases

1. Numbers ending with 5

If a number is in the form of $n5$, the square of it is $n(n + 1) \mid 25$

Example $45^2 = 4 \times 5 \mid 25 = 2025$

$$135^2 = 13 \times 14 \mid 25 = 18{,}225$$

This is nothing but the application of the multiplication method using the sum of unit's digits.

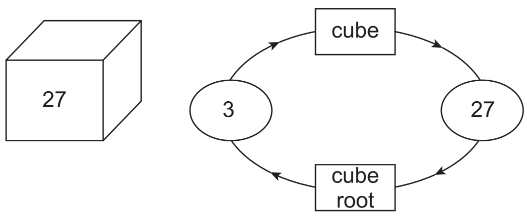
We can use this method to find out the squares fractions also like $1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}$, etc.

Process: Multiply the integral portion by the next higher integer and add $\frac{1}{4}$.

For example, $\left(6\frac{1}{2}\right) = 6 \times 7 + \frac{1}{4} = 42\frac{1}{4}$

CUBING

A number whose cube root is an integer is called a perfect cube.



Properties of a Cube

1. The sum of the cubes of any number of consecutive integers starting with 1 is the square of some integer. (e.g., $1^3 + 2^3 = 9 = 3^2$, $1^3 + 2^3 + 3^3 = 36 = 6^2$, etc.)
2. Unit digit of any cube can be any digit from 0 to 9.

Methods of Cubing

We can find the cube of any number close to a power of 10 say 10^n with base $= 10^n$ by finding the surplus or the deficit (x). The answer will be obtained in three parts. $B + 3x \mid 3 \cdot x^2 \mid x^3$

The left two parts will have n digits.

104^3

Base B = 100 and surplus $= x = 4$

$(100 + 3 \times 4)|3 \times 4^2|4^3 = 112|48|64 = 11{,}24{,}864$

109^3

Base B = 100 and $x = 9$

$(100 + 3 \times 9)|3 \times 9^2\ |9^3 = 127|243|729 = 12{,}95{,}029$

98^3

Base B = 100 and $x = -2$

$(100 - 3 \times 2) \mid 3 \times (-2)^2 \mid (-2)^3 = 94 \mid 12 \mid -8 = 94 \mid 11 \mid 100 - 8 = 941{,}192$

VEDIC MATHS TECHNIQUES IN ALGEBRA

1. If one is in ratio, the other one is zero

This formula is often used to solve simple simultaneous equations that may involve big numbers. However, these equations in special cases can be visually solved because of a certain ratio between the co-efficients. Consider the following example:

$$6x + 7y = 8$$

$$19x + 14y = 16$$

Here, the ratio of co-efficients of y is the same as that of the constant terms. Therefore, the 'other' is zero, i.e., $x = 0$. Hence, the solution of the equations is $x = 0$ and $y = 8/7$.

Alternatively,

$19x + 14y = 16$ is equivalent to $(19/2)x + 7y = 8$.

Therefore, x has to be zero and no ratio is needed; just divide by 2!

Note that it would not work if both had been 'in ratio':

$$6x + 7y = 8$$

$$12x + 14y = 16$$

This formula is easily applicable to more general cases with any number of variables. For instance,

$$ax + by + cz = a$$

$$bx + cy + az = b$$

$$cx + ay + bz = c$$

which yields $x = 1$, $y = 0$, and $z = 0$

2. When samuccaya is the same, then that samuccaya is zero

Consider the following symbols: N_1 – Numerator 1, N_2 – Numerator 2, D_1 – Denominator 1, D_2 – Denominator 2 and so on.

This formula is useful for solving equations that can be solved visually. The word 'samuccaya' has various meanings in different applications. For instance, it may mean a term, which occurs as a common factor in all the terms concerned. For example, an equation '$12x + 3x = 4x + 5x$'. Since 'x' occurs as a common factor in all the terms, therefore, $x = 0$ is the solution. Alternatively, samuccaya is the product of independent terms. For instance, in $(x + 7)(x + 9) = (x + 3)(x + 21)$, the samuccaya is $7 \times 9 = 3 \times 21$; therefore, $x = 0$ is the solution. It is also the sum of the denominators of two fractions having the same numerical numerator, for example:

$1/(2x - 1) + 1/(3x - 1) = 0$ means $5x - 2 = 0$

The more commonly used meaning is 'combination' or total. For instance, if the sum of the numerators and the sum of denominators are the same, then that sum is zero. Therefore,

$$\frac{2x+9}{2x+7} = \frac{2x+7}{2x+9}$$

Therefore, $4x + 16 = 0$ or $x = -4$

This meaning ('total') can also be applied in solving the quadratic equations. The total meaning not only imply sum but also subtraction. For instance, when given $N_1D_1 = N_2/D_2$, if $N_1 + N_2 = D_1 + D_2$ (as shown earlier), then this sum is zero. Mental cross multiplication reveals that the resulting equation is quadratic (the co-efficients of x^2 are different on the two sides). So, if $N_1 - D_1 = N_2 - D_2$, then that samuccaya is also zero. This yields the other root of a quadratic equation.

The interpretation of 'total' is also applied in multi-term RHS and LHS. For instance, consider

$$\frac{1}{x-7} + \frac{1}{x-9} = \frac{1}{x-6} + \frac{1}{x-10}$$

Here, $D_1 + D_2 = D_3 + D_4 = 2x - 16$. Therefore, $x = 8$.

There are several other cases where samuccaya can be applied with great versatility. For instance, 'apparently cubic' or 'biquadratic' equations can be easily solved as shown below:

$$(x - 3)^2 + (x - 9)^3 = 2(x - 6)^3$$

Note that $x - 3 + x - 9 = 2(x - 6)$. Therefore, $(x - 6) = 0$ or $x = 6$.

Consider

$$\frac{(x+3)^3}{(x+5)^3} = \frac{x+1}{x+7}$$

Observe: $N_1 + D_1 = N_2 + D_2 = 2x + 8$

Therefore, $x = -4$

CHAPTER

2

Number System*

LEARNING OBJECTIVES

After completion of this chapter, the reader should be able to understand:

- Numbers and their different types
- Definitions and properties of these numbers
- Concepts related to these numbers
- Different types of questions that are covered in the CAT
- Methods for solving these questions

INTRODUCTION

Here, we will be discussing the relative importance of various concepts in the number system with respect to CAT preparation as it has been one of the important topics in QA historically. From the past 15 years, CAT paper consisted of questions (almost 20%) from the number system. However, in this chapter, logic has an important role to play than the numbers. In other words, we can say that the logical processes take precedence over calculations in finding solution to exceptionally complex mathematical problems in number system. Students are expected to have a clear understanding of the definitions as well as concepts and develop a keen insight on numbers and their properties. Apart from these skills, they should try to maximize the potential of learning and solving every question.

The questions are asked based on the following two approaches:

1. ***Definitions and properties of numbers:*** In this section, questions will be based upon the definitions of different kinds of numbers. Apart from this, questions can be asked from some of the very basic calculations, formula, or properties of numbers.
2. ***Concepts:*** Some of the concepts on which questions are being asked are as follows:
   (a) LCM and HCF
   (b) Divisibility rules (for base 10)
   (c) Number of divisors
   (d) Number of exponents
   (e) Remainders
   (f) Base system
   (g) Units digit
   (h) Tens digit
   (i) Pigeonhole principle

CLASSIFICATION OF NUMBERS/ INTEGERS

Natural Numbers

Natural numbers are counting numbers, that is, the numbers that we use to count any number of things. For example, 1, 2, 3, ….

The lowest natural number is 1.

Whole Numbers

When zero is included in the list of natural numbers, then they are known as whole numbers. For example, 0, 1, 2, …

The lowest whole number is 0.

Integers

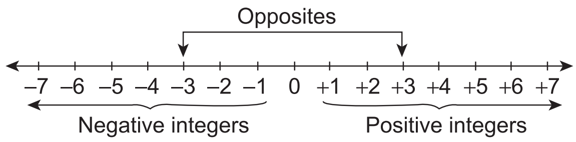
Integers are whole numbers, negative of whole numbers, and zero. For example, 43, 434, 235, 28, 2, 0, −28, and −3030

*To read more about Number System, you may use the book "Number System For CAT" by Nishit Sinha.

are integers; however, numbers such as 1/2, 4.00032, 2.5, Pi, and –9.90 are not whole numbers.

Number Line

The number line is used to represent the set of real numbers. The following is the brief representation of the number line:



Properties of Number Line

1. The number line goes till infinity in both directions. This is indicated by the arrows.
2. Integers greater than zero are called positive integers. These numbers are to the right of zero on the number line.
3. Integers less than zero are called negative integers. These numbers are to the left of zero on the number line.
4. The integer zero is neutral. It is neither positive nor negative.
5. The sign of an integer is either positive (+) or negative (–), except zero, which has no sign.
6. Two integers are opposites if each of them is at the same distance from zero, but on opposite sides of the number line. One will have a positive sign, and the other will have a negative sign. In the abovementioned number line, +3 and –3 are labelled as opposites. In other words, the whole negative number scale looks like a mirror image of the positive number scale, with a number like –15 being the same distance away from 0 as for number 15.
7. The number halfway between –1 and –2 is –1.5 and the number half way between 1 and 2 is 1.5.
8. We represent positive numbers without using a positive sign. For example, we would write 29.1 instead of +29.1. However, when we talk of negative numbers, the sign must be present.

Prime Numbers and Composite Numbers

Prime Numbers

Among natural numbers, we can distinguish prime numbers and composite numbers.

All the numbers that are divisible by 1 and itself only are known as prime numbers.

As mentioned earlier, primes can be natural numbers only. In other words, we can say that all the numbers that have only two factors are known as prime numbers. Prime numbers can also be seen as the building blocks. Further, we combine two or more than two or same or distinct prime numbers to create numbers larger than these prime numbers,

For example, $3 \times 2 = 6$

The following is the list of all prime numbers that are less than 1000.

2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61
67	71	73	79	83	89	97	101	103	107	109	113	127	131	137	139	149	151
157	163	167	173	179	181	191	193	197	199	211	223	227	229	233	239	241	251
257	263	269	271	277	281	283	293	307	311	313	317	331	337	347	349	353	359
367	373	379	383	389	397	401	409	419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541	547	557	563	569	571	577	587	593
599	601	607	613	617	619	631	641	643	647	653	659	661	673	677	683	691	701
709	719	727	733	739	743	751	757	761	769	773	787	797	809	811	821	823	827
829	839	853	857	859	863	877	881	883	887	907	911	919	929	937	941	947	953
967	971	977	983	991	997												

Using the following table, we can find number of prime numbers between every 100 numbers.

Numbers from-to	1–100	101–200	201–300	301–400	401–500	501–600	601–700	701–800	801–900	901–1000
Number of primes	25	21	16	16	17	14	16	14	15	14

Today, the largest known prime number is 78,16,230-digit prime number $2^{259,64,951} - 1$. It was found in early 2005; but, how big have the 'largest known primes' been historically?, and when might we see the first billion-digit prime number?

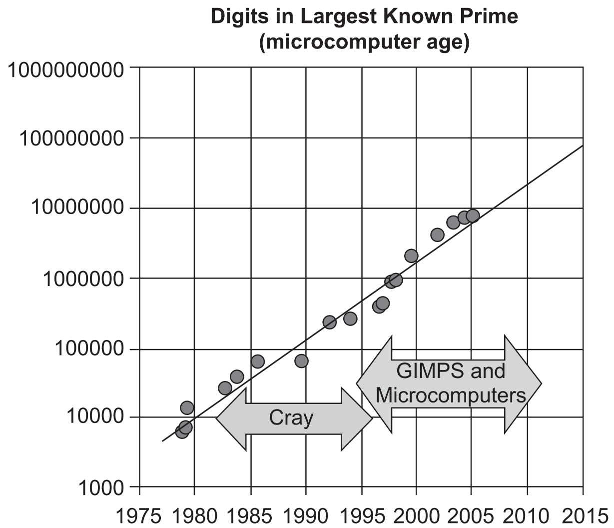
The following are the records that were used before the invention of electronic computers.

Number	Digits	Year	Prover	Method
$2^{17} - 1$	6	1588	Cataldi	Trial division
$2^{19} - 1$	6	1588	Cataldi	Trial division
$2^{31} - 1$	10	1772	Euler	Trial division
$(2^{59} - 1)/179,951$	13	1867	Landry	Trial division
$2^{127} - 1$	39	1876	Lucas	Lucas sequences
$(2^{148} + 1)/17$	44	1951	Ferrier	Proth's theorem

Until 1951, the prime number found by Lucas in 1876 was accepted as the largest prime number. In 1951, Ferrier used a mechanical desk calculator and techniques that are based on partial inverses of Fermat's little theorem (see the pages on remainder theorem). Using these techniques, he slightly improved this record by finding a 44-digit prime number.

Thus, in 1951, Ferrier found the prime $(2^{148} + 1)/17 =$ 20988936657440586486151264256610222593863921.

When will we have a one billion-digit prime number?



However, this record was very short-lived. In the same year (i.e., 1951), the advent of electronic computers helped human being in finding a bigger prime number than the earlier one.

In 1951, Miller and Wheeler began the electronic computing age by finding several primes and they had a new 79-digit prime number: $2^{127} - 1$. In addition, we know that this was the computer age and everybody was working hard to find the primes with the help of computers. Therefore, records were broken with a never-before pace.

Can we have a single formula representing all the prime numbers?

Until now, all the attempts done in this regard have proved to be fruitless. This is because there is no symmetricity between the differences among the prime numbers. Sometimes, two consecutive prime numbers differ by 2, 4, 10,000 or more. Therefore, there is no standard formula that can represent the prime numbers.

However, there are some standard notations that give us limited number of prime numbers: $N^2 + N + 41$. For all the values of N from −39 to +39, this expression gives us a prime number. Another similar example is $N^2 + N + 17$.

It is necessary to remember that all the prime numbers (>3) are of the form $6n \pm 1$ form (where n is any natural number); that is, all the prime numbers (>3) when divided by 6 give either 1 or 5 as the remainder.

> **Note:** Here, it is important to know that if a number gives a remainder of 1 or 5 when divided by 6, it is not necessarily a prime number. For example, 25 when divided by 6 gives remainder = 1; however, 25 is not a prime number.

Composite Numbers

A number is composite if it is the product of two or more than two distinct or same prime numbers. For example, 4, 6, 8,....

$4 = 2^2$

$6 = 2^1 \times 3^1$

The lowest composite number is 4.

All the composite numbers will have at least 3 factors.

Even and Odd Numbers

Let us assume N as an integer. If there exists an integer P such that $N = 2P + 1$, then **N is an odd number**. If there exists an integer P such that $N = 2P$, then **N is an even number**.

In simple language, even numbers are those integers that are divisible by 2 and odd numbers are those integers that are not divisible by 2. Even and odd numbers can be positive as well as negative also.

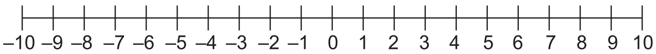
In other words, if x is an integer (even or not), then $2x$ will be an even integer; this is because it is a multiple of 2. Further, x raised to any positive integer power will be an even number, and therefore, x^2, x^3, x^4, etc., will be even numbers.

Any integer that is not a multiple of 2 is called an *odd* number. For instance, −1, 3, 6883, and −8147 are all odd numbers. Any odd number raised to a positive integer power will also be an odd number, and therefore, if x is an odd number, then x^2, x^3, x^4, etc., will be odd numbers.

The concept of even and odd numbers are most easily understood in the binary base. The abovementioned definition simply states that even numbers end with a 0 and odd numbers end with a 1.

Comparing Integers

We can compare two different integers by looking at their positions on the number line. For any two different places on the number line, the integer on the right-hand side is greater than the integer on the left-hand side. Note that every positive integer is greater than any negative integer.



For example, $9 > 4$, $6 > -9$, $-2 > -8$, and $0 > -5$, $-2 < 1$, $8 < 11$, $-7 < -5$, and $-10 < 0$

Remember

1. 1 is neither prime nor composite.
2. 0 is neither positive nor negative.

Example 1 Two of a, b, c, and d are even and two are odd, not necessarily in order. Which of the following is definitely even?

(a) $a + b + c - 2d$ (b) $a + 2b - c$
(c) $a + b - c + d$ (d) $2a + b + c - d$

Solution Since we do not know which two are even and two are odd, we will have to do trial-and-error method to solve this problem using the given options.

In option (a), if a and b are even, and c and d are odd, then this will lead us to odd number.

In option (b), if a and b are even, and c is odd, then this will lead us to odd number.

In option (d), if a and b are odd, and c and d are even, then this will lead us to odd number.

In option (c), whatever is the value of a, b, c, and d, it is always going to be an even number.

Thus, we can say that any type of calculation done with two even and two odd numbers will always result in an even number. Hence, the answer is option (c).

Example 2 If N, $(N + 2)$, and $(N + 4)$ are prime numbers, then the number of possible solutions for N is/are **(*CAT 2003*)**

(a) 1 (b) 2
(c) 3 (d) None of these

Solution There is only one triplet of prime numbers where difference between any two prime number is 2, that is, 3, 5, and 7. Therefore, $N = 3$ is the only solution. Hence, answer is (a).

Proof We know that the prime numbers are of the form $6M \pm 1$ (except 2 and 3). Now, if N is of the format $(6M + 1)$, then $(N + 2)$ will be of $(6M + 3)$ format and $(N + 4)$ will be of $(6M + 5)$ format. From these three numbers, since $(N + 2)$ is of $(6M + 3)$ format, it will be divisible by 3.

Similarly, if N is of the format $(6M - 1)$, then $(N + 2)$ will be of $(6M + 1)$ format and $(N + 4)$ will be of $(6M + 3)$ format. From these three numbers, since $(N + 4)$ is of $(6M + 3)$ format, it will be divisible by 3.

In both the cases, we find that one number that is given three numbers and it is divisible by 3. In the abovementioned example, (3, 5, and 7), one of the given three numbers is divisible by 3.

Example 3 Let x and y be the positive integers such that x is prime and y is composite. Then, which of the following is true? **(*CAT 2003*)**

(a) $y - x$ cannot be an even integer.
(b) xy cannot be an even integer $x + y$
(c) $\dfrac{x + y}{x}$ cannot be an even integer.
(d) None of these

Solution Eliminating the options,

To eliminate option (a): If $y = 4$ and $x = 2$, then $y - x$ can be even.

To eliminate option (b): If $y = 4$ and $x = 2$, then yx can be even.

To eliminate option (c): If $y = 6$ and $x = 2$, then it can also be even.

Therefore, answer is option (d).

QUESTIONS BASED UPON CONCEPTS

LCM

A common multiple is a number that is a multiple of two or more than two numbers. The common multiples of 3 and 4 are 12, 24, ...

The least common multiple (LCM) of two numbers is the smallest positive number that is a multiple of both.

Multiples of 3 — 3, 6, 9, 12, 15, 18, 21, 24, ...

Multiples of 4 — 4, 8, 12, 16, 20, 24, 28, ...

Therefore, LCM of 3 and 4 will be 12, which is the lowest common multiple of 3 and 4.

First of all, the basic question which lies is—for what kind of numbers, we can use LCM?

Let us explain it through an example: LCM of 10, 20, and 25 is 100. It means that 100 is the lowest number, which is divisible by all these three numbers.

Since (–100) is lower than 100 and divisible by each of 10, 20, and 25, can the LCM be (–100)? or can it be 0?

Further, what will be the LCM of (–10) and 20? Will it be (–20) or (–200) or (–2000) or smallest of all the numbers, that is, <–∞?

Answer to all these questions is very simple: LCM is a concept defined only for positive numbers, whether the number is an integer or a fraction. In other words, **LCM is not defined for negative numbers or zero**.

Now, we will define a different method for finding the LCM of two or more than two positive integers.

Process to Find LCM

Step 1 Factorize all the numbers into their prime factors.
Step 2 Collect all the distinct factors.
Step 3 Raise each factor to its maximum available power and multiply.

Example 4 LCM of 10, 20, 25.

Solution

Step 1 $10 = 2^1 \times 5^1$
$20 = 2^2 \times 5^1$
$25 = 5^2$
Step 2 2, 5
Step 3 $2^2 \times 5^2 = 100$

One of the principal advantage of using this method is that we can find the LCM of any number of numbers in a straight line without using the conventional method. The following explains this using the previous example:

The LCM of 10 and 20 = 20, and LCM of 20 and 25 = 100 (For this, we have to know which factor of 25 is not present in 20; then, we need to multiply it by this factor. Therefore, 25 is having 5^2 and 20 is having 5^1 only, and hence, we will multiply 20 by 5.)

Example 5 LCM of 35, 45, 55.

Solution First, let us determine the LCM of 35 and 45.

Now, $35 = 5^1 \times 7^1$ and $45 = 3^2 \times 5^1$.

Therefore, it can be observed here that 35 is not having 3^2 in it, and hence, we will multiply 35 by 3^2.

Thus, LCM of 35 and 45 = 35×3^2 (Further, you can start with 45 to find the missing factors of 35 in 45.)

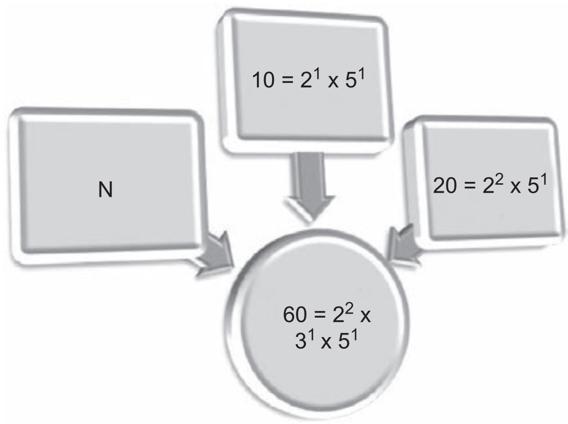
Now, we will find the LCM of 35×3^2 and $55 = 5^1 \times 11^1$

$55 = 5^1 \times 11^1$

Now, 11^1 is not present in 35×3^2. Therefore, we will multiply 35×3^2, and finally, LCM = $35 \times 3^2 \times 11^1 = 3465$.

Example 6 LCM of three natural numbers 10, 20, and $N = 60$. How many values of N are possible?

Solution We have already discussed that to generate the LCM, we must multiply the prime numbers with the highest available power. Therefore, let us start with factorizing the number.



$2^2 \times 5^1$ is already present in 20; however, 3 is not present in either 10 or 20. Therefore, we can conclude that 3^1 has to be from N. This is the minimum value of $N = 3$. Second, we can also say that N may contain powers of 2 and 5 as long as the maximum power of 2 = 2 and maximum power of 5 = 1 (as in $2^2 \times 5^1$).

Therefore, the total different values of $N = (3^1 \times 2^0 \times 5^0)$, $(3^1 \times 2^1 \times 5^0)$, $(3^1 \times 2^2 \times 5^0)$, $(3^1 \times 2^0 \times 5^1)$, $(3^1 \times 2^1 \times 5^1)$, $(3^1 \times 2^2 \times 5^1)$ = 3, 6, 12, 15, 30, 60 = 6 values

Highest Common Factor (HCF)

The factors that are positive integral values of a number and can divide that number is called HCF. HCF, which is also known as Greatest Common Divisor (GCD), is the highest value that can divide the given numbers.

Factors of 20 — 1, 2, 4, 5, 10, 20.
Factors of 30 — 1, 2, 3, 5, 6, 10, 15, 30.
Therefore, 10 will be the HCF of 20 and 30.

Process to Find HCF

Step 1 Factorize all the numbers into their prime factors.
Step 2 Collect all the common factors.
Step 3 Raise each factor to its minimum available power and multiply.

Example 7 HCF of 100, 200, and 250

Solution

Step 1 $100 = 2^2 \times 5^2$
$200 = 2^3 \times 5^2$
$250 = 5^3 \times 2^1$
Step 2 2, 5
Step 3 $2^1 \times 5^2 = 50$

Alternatively, to find HCF of numbers such as 100, 200, and 250, we have to observe the common quantity that can be taken from these numbers. To do this, we can write these numbers as $(100x + 200y + 250z)$, and now, it can be very easily observed that we can take 50 as the common number from the given numbers.

The LCM and HCF can be summarized as follows: it is very essential to understand the mechanism of determining LCM and HCF. These two concepts can be understood easily by the following example:

Example 8 Find the LCM and HCF of 16, 12, 24.

Solution

No.	Multiples	Factors
16	16, 32, 48, 64, 80, 96, 112, 128, ...	1, 2, 4, 8, 16
12	12, 24, 36, 48, 60, 72, 84, 96, 108, ...	1, 2, 3, 4, 6, 12
24	24, 48, 72, 96, 120, 144, 168, 192, ...	1, 2, 3, 4, 6, 8, 12, 24

Common Multiple	Common Factor
48	1, 2, 3, 4
Lowest common multiple	Highest common factor
48	4

The standard formulae are as follows:

1. LCM × HCF = product of two numbers.
   This formula can be applied only in the case of two numbers. However, if the numbers are relatively prime to each other (i.e., HCF of numbers = 1), then this formula can be applied for any number of numbers.
2. LCM of fractions = LCM of numerator of all the fractions/HCF of denominator of fractions.
3. HCF of fractions = HCF of numerator of all the fractions/LCM of denominator of fractions.
4. HCF of (sum of two numbers and their LCM) = HCF of numbers.

Example 9 HCF of two natural numbers A and B is 120 and their product is 10,000. How many sets of values of A and B is/are possible?

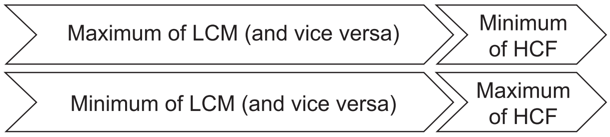
Solution HCF $(A, B) = 120 \Rightarrow 120$ is a common factor of both the numbers (120 being the HCF). Hence, 120 is present in both the numbers. Therefore, the minimum product of A and $B = 120 \times 120 = 14{,}400$. Hence, no set of A and B are possible for satisfying the conditions.

Maxima and Minima in LCM/HCF

If the product of two numbers is given, and none of LCM or HCF is given, then this gives rise to the case of maxima and minima.

Primarily, we use the formula LCM × HCF = product of two numbers. Although this formula only provides the basic framework, we need to visualize the situation to solve these questions.

By using the formula, LCM × HCF = product of two number, we can say that since RHS is constant, LHS will be inversely proportional to HCF (subject to the values being natural numbers).



Example 10 Product of two natural numbers = 144. What is the (a) largest possible and (b) smallest possible HCF of these two natural numbers?

Solution Let us first factorize $144 = 12 \times 12$

$$= (2^2 \times 3) \times (2^2 \times 3)$$

The largest possible HCF occurs when LCM = HCF and numbers are equal.

We already know that product of two natural numbers = LCM × HCF.

Since numbers have to be equal, each of the numbers = 12, and the largest possible HCF = 12.

Therefore, the smallest possible HCF has to be equal to 1 (possible set of numbers = 144, 1).

Example 11 Product of two natural numbers = 144. How many different values of LCM are possible for these two natural numbers?

Solution We have already seen in the abovementioned question that the largest possible value of HCF = 12 and consequently, small values of LCM = 12.

Let us see the different values of HCF and corresponding LCM.

Therefore, total different values of LCM = 6

HCF = 12	HCF = 6	HCF = 4	HCF = 3	HCF = 2	HCF = 1
LCM = 12	LCM = 24	LCM = 36	LCM = 48	LCM = 72	LCM = 144

The following gives some questions based on the standard application of LCM and HCF:

Case I Time and Work

Example 12 Tatto can do a work in 10 days and Tappo can do the same work in 12 days. How many days will it take if both start working together?

Solution Let us assume total work = LCM of (10, 12) units = 60 units. Now, 60 units of work is being done by Tatto in 10 days and Tatto is doing 6 units of work per day; similarly, Tappo is doing 5 units of work per day. Hence, they are doing 11 units of work in one day together.

Thus, they will take $\frac{60}{11} = 5\frac{5}{11}$ days to complete the work.

Case II Time, speed, and distance — circular motion

Example 13 The speed of A is 15 m/s and speed of B is 20 m/s. They are running around a circular track of length 1000 m in the same direction. Let us find after how much time, will they meet at the starting point if they start running at the same time.

Solution Time taken by A and B in taking one circle are 66.66 s and 50 s, respectively. Therefore, LCM (66.66, 50) = 20s.

Case III Number system— tolling the bell

Example 14 There are two bells in a temple. Both the bells toll at a regular interval of 66.66 s and 50 s, respectively. After how much time, will they toll together for the first time?

Solution Time taken by Bell 1 and Bell 2 to toll is 66.66 s and 50 s. Therefore, LCM (66.66, 50) = 200 s.

Here, it can be observed that the mathematical interpretation of both the questions are same, only the language has been changed.

Case IV Number System— Number of Rows

Example 15 There are 24 peaches, 36 apricots, and 60 bananas and they have to be arranged in several rows in such a way that every row contains same number of fruits of one type. What is the minimum number of rows required for this arrangement?

Solution We can arrange one fruit in one row, and still in (24 + 36 + 60) = 120 rows, we can arrange all the fruits. Further, even we can arrange two fruits in one row and can arrange all the fruits in 60 rows. However, for the rows to be minimum, the number of fruits should be maximum in one row.

HCF of 24, 36, 60 = 12, and therefore, 12 fruits should be there in one row.

Hence, the number of rows = 10

Case V Number System— finding remainder

Example 16 Find the lowest three-digit number that when divided by 4 and 5 gives 3 as the remainder.

Solution Let us assume that there is no remainder. Therefore, the number has to be a multiple of LCM of 4 and 5. Now, LCM (4, 5) = 20

However, there is a remainder of 3 when divided by 4 and 5. Therefore, the number will be in the form of (207N + 3).

Hence, numbers are 23, 43, 63, 83, 103, and so on.

Thus, the three-digit number is 103.

Divisibility Rules (For Decimal System)

Divisibility rules are quite imperative. This is because using this, we can infer if a particular number is divisible by other number or not, without actually dividing it.

Divisibility rules of numbers are specific to that particular number only. It simply means that divisibility rules of different numbers will be different. We shall now see a list of divisibility rules for some of the natural numbers.

Divisibility Rules

For 2 If units digit of any number is 0, 2, 4, 6 or 8, then that number will be divisible by 2.

For 3 If sum total of all the digits of any number is divisible by 3, then the number will be divisible by 3 (e.g., 123, 456, etc.)

Example 17 How many values of A are possible if 3245684 A is divisible by 3?

Solution Sum total of the number = 32 + A

For this number to be divisible by 3, A can take three values namely 1 or 4 or 7. (No other values are possible since A is the units digit of the number.)

For 4 If the last two digit of a number is divisible by 4, then that number will be divisible by 4 (e.g., 3796, 248, 1256, etc.)

For 5 If the last digit of the number is 5 or 0, then that number will be divisible by 5.

For 6 If the last digit of the number is divisible by two and sum total of all the digits of number is divisible by 3, then that number will be divisible by 6.

For 7 The integer is divisible by 7 if and only if the difference of the number of its thousands and the remainder of its divisible by 1000 is divisible by 7.

Example: Let us take the number 795. The units digit is 5 and when it is doubled, we get 10. The remaining part of the number (i.e., the tens) is 79. If 10 is subtracted from 79, we get 69. Since this result is not divisible by 7, the original number 695 is also not divisible by 7.

For 8 If the last 3 digits of number is divisible by 8, then the number itself will be divisible by 8 (e.g., 128, 34568, 76232, etc).

For 9 If the sum of digits of the number is divisible by 9, then the number will be divisible by 9 (e.g., 1,298,35,782).

1 + 2 + 9 + 8 + 3 + 5 + 7 + 8 + 2 = 45. Since 45 is divisible by 9, number will be divisible by 9.

If units digit of any number is 0, 2, 4, 6, or 8, then that number will be divisible by 2.

Example 18 How many pairs of A and B are possible in number 89765A4B if it is divisible by 9, given that the last digit of number is even?

Solution Sum of the digits of number is 8 + 9 + 7 + 6 + 5 + A + 4 + B = 39 + A + B.

Therefore, $(A + B)$ should be 6 or 15. Next value should be 24; since A and B are digits, so it cannot be more than 18. Possible pairs of A and B are as follows:

A	B
0	6
1	5
2	4
3	3
4	2
5	1
6	0
7	8
8	7
9	6
6	9

Since B is even, six possible set of values of A and B are there.

For 11 A number is divisible by 11, if the difference between the sum of the digits at even places and the sum of the digits at odd places is divisible by 11 (zero is divisible by 11).
For example**,** 65,95,149 is divisible by 11 as the difference of $6 + 9 + 1 + 9 = 25$ and $5 + 5 + 4 = 14$ is 11.

For 12 If the number is divisible by 3 and 4, then the number will be divisible by 12 (e.g., 144, 348).

For 13 $(A + 4B)$, where B is the units place digit and A is all the remaining digits.

For example, let us check the divisibility of 1404 by 13. Here, $A = 140$ and $B = 4$, then $A + 4B = 140 + 4 \times 4 = 156$. This 156 is divisible by 13, and therefore, 1404 will be divisible by 13.

For 14 If the number is divisible by both 2 and 7, then the number will be divisible by 14.

For 15 A number is divisible by 15, if the sum of the digits is divisible by 3 and units digit of the number is 0 or 5.

For example**,** 225, 450, 375, etc.

For 16 A number is divisible by 16, if the number formed by the last 4 digits of the given number is divisible by 16.

For example, 125,78,320 is divisible by 16, since the last 4 digits of the number 8320 is divisible by 16.

For 17 $(A - 5B)$ where B is the unit's place digit and A is all the remaining digits.

For 18 Number should be divisible by both 9 and 2.

For 19 $(A + 2B)$ where B is the unit's place digit and A is all the remaining digits.

If the sum of the number of tens in the number and twice the units digit is divisible by 19, then the number is divisible by 19.

For example, let us take 665. The units digit is 5 and when it is doubled, we get 10. The remaining part of the number is 66. If 10 (which is the units digit doubled) is added to 66, we get 76. Since this result (76) is divisible by 19, it means the original number 665 is also divisible by 19.

For 20 Number should be divisible by 4 and 5.

The process to find the divisibility rule for prime numbers is simple; however, it is difficult to express in words. Let us discuss it in the following.

We are creating the divisibility rule for P, a prime number.

Step 1 Find the multiple of P, closest to any multiple of 10. (This will be essentially of the form $(10K + 1)$ or $(10K - 1)$.)

Step 2 If it is $(10K - 1)$, then the divisibility rule will be $(A + KB)$, and if it is $(10K + 1)$, then the divisibility rule will be $(A - KB)$, where B is the unit's place digit and A is all the remaining digits.
For example, let us find the divisibility rule of 23: Lowest multiple of 23, which is closest to any multiple of 10 = 69 = $(7 \times 10 - 1)$

Therefore, rule is $(A + 7B)$.

Number of Divisors

If one integer can be divided by another integer an exact number of times, then the first number is said to be a multiple of the second, and the second number is said to be a factor of the first.

For example, 48 is a **multiple** of 6 because it can divide 48 an exact number of times (in this case, it is 8 times). In other words, if you have 48 apples, we can distribute them among 6 persons equally without splitting any apple.

Similarly, 6 is **a factor** of 48. On the other hand, 48 is not a multiple of 5, because 5 cannot divide 48 an exact number of times. Therefore, 5 is not a factor of 48.

When we talk about number of divisors of any number, we are talking about positive integral divisor of that number.

For example, it can be observed that 20 has six divisors, namely 1, 2, 4, 5, 10, and 20.

Formation of Divisors

$$20 = 2^2 \times 5^1$$

Now, let us assume that 20 will be divisible by which numbers:

$\frac{2^2 \times 5^1}{7}$ Yes/No

$\dfrac{2^2 \times 5^1}{2^1}$ Yes/No

$\dfrac{2^2 \times 5^1}{2^3}$ Yes/No

$\dfrac{2^2 \times 5^1}{2^1 \times 5^1}$ Yes/No

Answer to the abovementioned posers can be given in the following order—No, Yes, No, Yes.

We can observe that the denominator should have powers of only 2 and 5—powers of 2 should be from 0 to 2 and powers of 5 should be 0 to 1.

$$\frac{2^2 \times 5^1}{2^{0-2} \times 5^{0-1}}$$

Hence, we will take three powers of 2, that is, 2^0, 2^1, and 2^2 and two powers of 5, that is, 5^0 and 5^1.

Divisors will come from all the possible arrangements of powers of 2 and 5.

$$2^0 \times 5^0 = 1$$

$$2^0 \times 5^1 = 5$$

$$2^1 \times 5^0 = 2$$

$$2^1 \times 5^1 = 10$$

$$2^2 \times 5^0 = 4$$

$$2^2 \times 5^1 = 20$$

By summarizing these calculations, following formula can be derived:

If N is any number that can be factorized like $N = a^p \times b^q \times c^r \times ...$, where a, b, and c are prime numbers, then the number of divisors = $(p + 1)(q + 1)(r + 1)$

Example 19 Find the number of divisors of $N = 420$.

Solution $N = 420 = 2^2 \times 3^1 \times 7^1 \times 5^1$

Therefore, the number of divisors = $(2 + 1)(1 + 1)(1 + 1)(1 + 1) = 24$.

Example 20 Find the total number of even and prime divisors of $N = 420$.

Solution $N = 420 = 2^2 \times 3^1 \times 7^1 \times 5^1$

Odd divisors will come only if we take zero power of 2 (since any number multiplied by any power (≥1) of 2 will give us an even number). Odd divisors will come if we take $N_1 = 2^0 \times 3^1 \times 7^1 \times 5^1$

Number of odd divisors = $(0 + 1)(1 + 1)(1 + 1)(1 + 1) = 8$

Therefore, total number of even divisors = total number of divisors – number of odd divisors = 24 – 8 = 16

Alternatively, we can also find the number of even divisors of N as 420 (in general, for any number).

$$420 = 2^2 \times 3^1 \times 7^1 \times 5^1$$

To obtain the factors of 420, which are even, we will not consider 2^0, since $2^0 = 1$

Therefore, number of even divisors of 420 = $(2)(1 + 1)(1 + 1)(1 + 1) = 16$.

(We are not adding 1 in the power of 2, since we are not taking 2^0, that is, we are not taking one power of 2.)

Prime divisor = 4 (namely 2, 3, 5, and 7 only)

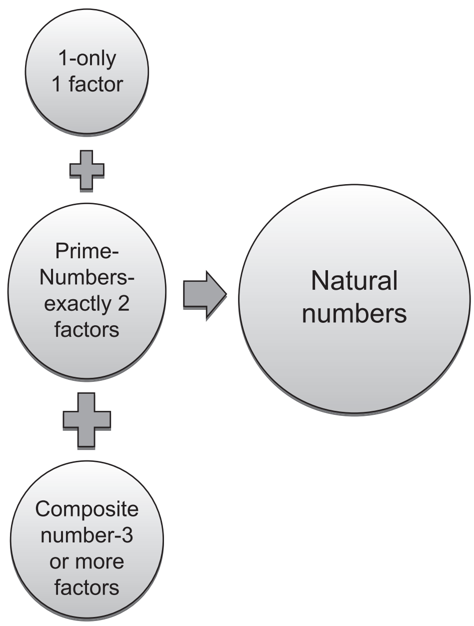
Example 21 $N = 2^7 \times 3^5 \times 5^6 \times 7^8$. How many factors of N are divisible by 50 but not by 100?

Solution All the factors that are divisible by 50 but not divisible by 100 will have at least two powers of 5, and one power of 2. Further, its format will be $2^1 \times 5^{2+y}$.

Therefore, the number of divisors = $1 \times 6 \times 5 \times 9 = 270$.

The following discusses the determination of prime factors and composite factors:

We know that natural number line (starting from 1, 2, 3, …) can be classified on the basis of number of factors to the natural number.



From the given graphics, we conclude the following:

(i) On the basis of number of factors, natural number line can be categorized into three parts: (a) 1, (b) prime number, and (c) composite factors.

(ii) Lowest composite number = 4.

The essence of the whole discussion lies in the fact that the total number of factors of any natural number = 1 (number 1 is a factor of all the natural numbers) + prime factors + composite factors.

Therefore, once we complete the prime factorization for finding the number of prime factors, we just need to count the number of prime factors. To calculate the number of composite factors, we will subtract the number of prime factors and 1 from the total number of factors.

Example 22 Find the number of prime factors and composite factors of $N = 420$.

Solution $420 = 2^2 \times 3^1 \times 5^1 \times 7^1$

Number of prime factors = 4 (namely 2, 3, 5, 7).

Total number of factors = $(2 + 1)(1 + 1)(1 + 1)(1 + 1) = 3 \times 2 \times 2 \times 2 = 24$

Therefore, the total number of composite factors = total number of factors – prime factors – 1 = 24 – 4 – 1 = 19.

The following discusses the determination of factors that are perfect squares or cubes or higher power: A number will be perfect square only if all the prime factors of this number will have even powers. Therefore, a number of the format 2^x will be a perfect square only if $x = 0, 2, 4, 6, 8$, etc.

Similarly, a number will be perfect cube only if all the prime factors of this number will have powers divisible by 3. Therefore, a number of the format 2^y will be a cube only if $x = 0, 3, 6, 9$, etc.

Example 23 How many factors of the number $N = 720$ will be (a) perfect square, (b) cube, and (c) both a perfect square and cube?

Solution $N = 720 = 2^4 \times 3^2 \times 5^1$

(a) For a factor of $N = 720$ to be a perfect square, it should have only the following powers of its prime factors:

Powers of 2	Powers of 3	Powers of 5
2^0	3^0	5^0
2^2	3^2	
2^4		

Number of powers of 2 = 3
Number of powers of 3 = 2
Number of powers of 5 = 1
Hence, the total number of factors of $N = 720$ that are perfect square = $3 \times 2 \times 1 = 6$

(b) For a factor of $N = 720$ to be a cube, it should have only the following powers of its prime factors:

Powers of 2	Powers of 3	Powers of 5
2^0	3^0	5^0
2^3		

Number of powers of 2 = 2
Number of powers of 3 = 1
Number of powers of 5 = 1
Hence, the total number of factors of $N = 720$ that are cubes = $2 \times 1 \times 1 = 2$.

(c) For a factor of $N = 720$ to be both a cube and a square, it should have only the following powers of its prime factors:

Powers of 2	Powers of 3	Powers of 5
2^0	3^0	5^0

Number of powers of 2 = 1
Number of powers of 3 = 1
Number of powers of 5 = 1
Hence, the total number of factors of $N = 720$ that are cubes = $1 \times 1 \times 1 = 1$

The condition for two divisors of any number *N* to be co-prime to each other can be explained as follows: Two numbers are said to be co-prime to each other if their HCF = 1. This can happen only if none of the factors of the first number (other than 1) is present in the second number and vice versa.

Let us see it for $N = 12$

Total number of factors of 12 = 6 (namely 1, 2, 3, 4, 6, 12). Now, if we have to find set of factors of this number that are co-prime to each other, we can start with 1.

Number of factors that are co-prime to 1 = 5 (namely, 2, 3, 4, 6, 12).

Next, the number of factors that are co-prime to 2 = 1 (namely 3)

Therefore, the total number of set of factors of 12 that are co-prime to each other = 6

Thus, we can induce that if we have to find the set of factors that are co-prime to each other for $N = a^p \times b^q$, it will be equal to $[(p + 1)(q + 1) - 1 + pq]$.

If there are three prime factors of the number, that is, $N = a^p \times b^q \times c^r$, then set of co-prime factors can be given by $[(p + 1)(q + 1)(r + 1) - 1 + pq + qr + pr + 3pqr]$

Alternatively, we can find the set of co-prime factors of this number by pairing up it first, and then finding the third factor.

Example 24 Find the set of co-prime factors of the number $N = 720$.

Solution $720 = 2^4 \times 3^2 \times 5^1$

Using the formula for three prime factors $[(p + 1)(q + 1)(r + 1) - 1 + pq + qr + pr + 3pqr]$,

we get $[(4 + 1)(2 + 1)(1 + 1) - 1 + (4 \times 2) + (2 \times 1) + (4 \times 1) + (3 \times 4 \times 2 \times 1)] = 67$

Alternatively, let us find the first for $2^4 \times 3^2 = [(4 + 1)(2 + 1) - 1 + (4 \times 2)] = 22$

Now, $p^{22} \times 5^1$ will give us $[(22 + 1)(1 + 1) - 1 + 22 \times 1] = 67$.

Sum of Divisors

We can find the sum of divisors similar to the number of divisors of any number. If *N* is any number that can be factorized like $N = a^p \times b^q \times c^r x$, where a, b, and c are prime numbers, then

$$\text{Sum of the divisors} = \frac{(a^{p+1}-1)(b^{q+1}-1)(c^{r+1}-1)}{(a-1)(b-1)(c-1)}$$

Remainders

Dividend = quotient × divisor + remainder

The basic framework of remainder are as follows:

1. If N is a number divisible by 7, it can be written as $7K = N$, where K is the quotient.
2. When N is divided by 7, remainder obtained is 3. Therefore, it can be written as $7K + 3 = N$, where K is the quotient.
3. When N is divided by 7, remainder obtained is 3 and it is equivalent of saying remainder obtained is (–4) when divided by 7. It can be understood that when N is divided by 7, remainder obtained is $3 = N$ is 3 more than a multiple of 7 => Therefore, N is 4 short of another multiple of 7. Therefore, remainder obtained = –4.
4. When N is divided by 8, different remainders can be obtained. They are 0, 1, 2, 3, 4, 5, 6, 7 (8 different remainders). Similarly, when it is divided by 5, remainders 0, 1, 2, 3, 4 (5 different remainders) are obtained.

Basics of Remainder

1. If any positive number A is divided by any other positive number B and if $B > A$, then the remainder will be A itself. In other words, if the numerator is smaller than the denominator, then the numerator is the remainder. For example,

   Remainder of $\frac{5}{12} = 5$

   Remainder of $\frac{21}{45} = 21$
2. Remainder should always be calculated in its actual form, that is, we cannot reduce the fraction to its lower ratio. For example,

   Remainder of $\frac{1}{2} = 1$

   Remainder of $\frac{2}{4} = 2$

   Remainder of $\frac{3}{6} = 3$

   It can be observed that despite all the fractions being equal, remainders are different in each case.

Example 25 What is the remainder when 5×10^5 is divided by 6×10^6?

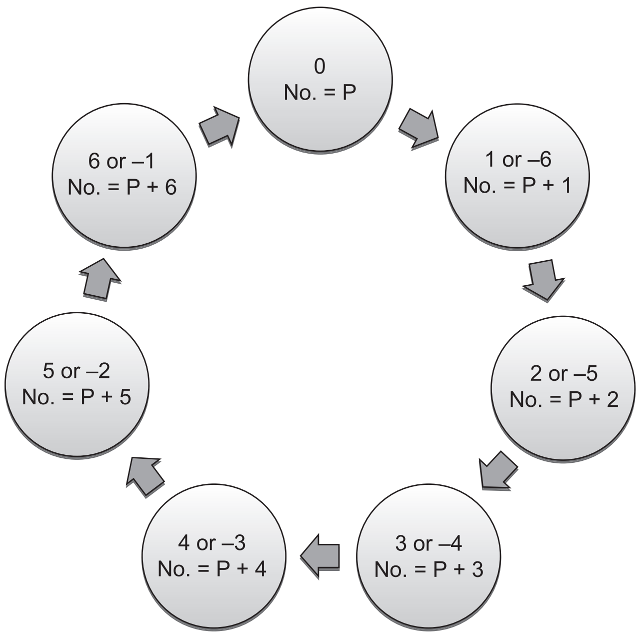
Solution As we know that we cannot reduce the fractions to its lower terms and numerator is less than denominator, the remainder obtained will be equal to 5×10^5.

3. The concept of negative remainder—as obvious from the name, this remainder implies that something has been left or something remains there. Therefore, remainder can simply never be negative. Its minimum value can be zero only and non-negative. For example, What is the remainder when –50 is divided by 7?

   Solution is $\frac{-50}{7} = \frac{-56+6}{7}$; this gives a remainder of 6

However, when we divide –50 by 7, we get –1 as the remainder. Now, since remainder has to be non-negative, we add 7 (quotient) to it that makes the final remainder as $(-1 + 7) = 6$. It can be explained in the following figure.

Let us assume that when P is divided by 7, remainder obtained as 0.



Therefore, when $(P + 1)$ will be divided by 7, remainder obtained will be either 1 or –6. Similarly, when $(P + 2)$ is divided by 7, the remainder obtained will be 2 or –5, and so on.

Now, there are two methods to find the remainder of any expression:

1. **Cyclicity method**
   for every expression of the remainder, there comes attached a specific cyclicity of remainders.

Example 26 What is the remainder when 4^{1000} is divided by 7?

Solution To find the cyclicity, we keep finding the remainders until some remainder repeats itself. It can be understood with the following example:

Number/7—	4^1	4^2	4^3	4^4	4^5	4^6	4^7	4^8
Remainder—	4	2	1	4	2	1	4	2

Now, 4^4 gives us the same remainder as 4^1; therefore, the cyclicity is of 3 (this is because remainders start repeating themselves after 4^3).

Thus, any power of 3 or a multiple of 3 will give a remainder of 1, and hence, 4^{999} will give 1 as the remainder. Final remainder = 4.

Example 27 What is the remainder when 4^{96} is divided by 6?

Solution Let us find the cyclicity.

Number/6—	4^1	4^2	4^3	4^4	4^5	4^6	4^7	4^8
Remainder—	4	4	4	4	4	4	4	4

In all cases, the remainder is 4, and therefore, the final remainder will be 4. Actually, it is not required to find remainders till 4^8 or even 4^3. 4^2 itself gives us a remainder of 4 when divided by 6, which is same as the remainder obtained when 4^1 is divided by 6. Therefore, the length of cycle = 1.

Hence, final remainder = 4. Further, it can be observed here that if we write $4^{100}/6 = 2^{200}/6 = 2^{199}/3$, then remainder obtained will be 2, which is not the right answer (as given in the CAT brochure of next year, i.e., CAT 2004.)

2. **Remainder Theorem Method**

The product of any two or more than two natural numbers has the same remainder when divided by any natural number as the product of their remainders.

Let us understand this through an example:

Example 28 Remainder $\frac{12 \times 13}{7}$ = Remainder $\frac{156}{7} = 2$

Solution The conventional way of doing this is Product →→→Remainder

Using the theorem method, we get

Remainder→→→

Product →→→ Remainder

Therefore, first, we will find the remainders of each individual number, and then, we will multiply these individual remainders to find the final remainder.

Remainder 12/7 = 5

Remainder 13/7 = 6

Remainder $\frac{12 \times 13}{7}$ = (5 × 6)/7 = Remainder 30/7 = 2

Example 29 What is the remainder obtained when (1421 × 1423 × 1425) is divided 12?

Solution Remainder of 1421/12 = 5

Remainder of 1423/12 = 7

Remainder of 1425/12 = 9

Remainder (1421 × 1423 × 1425)/12 = Remainder (5 × 7 × 9)/12 = Remainder (5 × 63)/12 = Remainder (5 × 3)/12 = 3

Successive Division

Let us assume that N is any number that is divided successively by 3 and 5. Here, we mean to say that at first, we divide N by 3, and then, the quotient obtained is divided by 5.

For example, let us consider the case where 50 is divided by 5 and 3 successively.

50 divided by 5 gives 10 as the quotient. Now, we will divide 10 by 3. Finally, it gives a quotient of 3 and remainder of 1.

Example 30 When a number N is divided successively by 3 and 5, remainder obtained are 1 and 2, respectively. What is the remainder when N is divided by 15?

Solution It can be seen that we are required to calculate it from back-end.

The family of numbers which when divided by 5 gives remainder as $2 = 5S + 2$

Therefore, $N = 3(5S + 2) + 1 = 15S + 7$

Now, if N is divided by 15, remainder = 7

Fermat's Remainder Theorem

Let P be a prime number and N be a number non-divisible by P. Then, remainder obtained when A^{P-1} is divided by P is 1.

(The remainder obtained when $\frac{A^{p-1}}{P} = 1$, if HCF $(A, P) = 1$.)

Example 31 What is the remainder when 2^{100} is divided by 101?

Solution Since it satisfies the Fermat's theorem format, remainder = 1.

Derivations

1. $\frac{(A+1)^N}{A}$ will always give 1 as the remainder (for all natural values of A and N).

Example 32 What is the remainder when 9^{100} is divided by 8?

Solution For $A = 8$, it satisfies the abovementioned condition. Therefore, remainder = 1.

Alternatively, we can apply either of cyclicity or theorem method to find the remainder.

2. $\frac{(A)^N}{A+1}$ when N is even, remainder is 1, and when N is odd, then remainder is A (for all natural values of A and N).

Example 33 What is the remainder when 2^{10} is divided by 3?

Solution Since N is even, remainder = 1

3. i. $(a^n + b^n)$ is divisible by $(a + b)$, if n is odd.

The extension of the abovementioned formula $(a^n + b^n + c^n)$ is divisible by $(a + b + c)$, if n is odd and a, b, and c are in arithmetic progression.

Example 34 What is the remainder obtained when $\frac{7^7 + 10^7 + 13^7 + 16^7}{46}$?

Solution It can be seen that 7, 10, 13, and 16 are in arithmetic progression and power n is odd. Further, denominator = 7 + 10 + 13 + 16 = 46. Hence, it will be divisible. Therefore, remainder obtained = 0.

Similarly, the abovementioned situation can be extended for any number of terms.

- $(a^n - b^n)$ is divisible by $(a + b)$, if n is even.
- $(a^n - b^n)$ is divisible by $(a - b)$, if n is even

Example 35 What is the remainder when $(15^{23} + 23^{23})$ is divided by 19? ***(CAT 2004)***

Solution It can be observed that $(15^{23} + 23^{23})$ is divisible by 38, and therefore, it will be divisible by 19 also. Hence, remainder = 0.

Alternatively, this problem can be done either by cyclicity method or theorem method.

Example 36 What is the remainder when $(16^3 + 17^3 + 18^3 + 19^3)$ is divided by 70? ***(CAT 2005)***

Solution We know that this is a basic multiplication and division question. However, using the abovementioned approach makes it very simple.

We know that $(a^n + b^n)$ is divisible by $(a + b)$, if n is odd. From this, we can say that $(a^n + b^n + c^n)$ is divisible by $(a + b + c)$, if n is odd, and similarly, $(a^n + b^n + c^n + d^n)$ is divisible by $(a + b + c + d)$. Now, (16 + 17 + 18 + 19) = 70, and therefore, remainder is zero.

The following table lists out different types of problems:

	Problem	Solution
1	Find the greatest number that will exactly divide a, b, and c	Required number = HCF of a, b, and c
2	Find the greatest number that will divide x, y, and z leaving remainders a, b, and c, respectively.	Required number (greatest divisor) = HCF of $(x - a)$, $(y - b)$ and $(z - c)$.
3	Find the least number that is exactly divisible by a, b, and c.	Required number = LCM of a, b, and c
4	Find the least number that when divided by x, y, and z leaves the remainders a, b, and c, respectively, and $(x - a) = (y - b) = (z - c) = N$	Required number = LCM of (x, y, and z) $- N$
5	Find the least number that when divided by x, y, and z leaves the same remainder r in each case.	Required number = (LCM of x, y, and z) + r

Units Digit

As discussed earlier, cyclicity exists for units digit of the numbers also. (However, it is necessary to remember that there is no relation between the cyclicity of remainders and the units digit.) Let us consider a simple example: $-2^5 = 32$. Here, we know that units digit of 2^5 is 2. However, problem occurs when we start taking large numbers like $25{,}678^{2345}$, and so on. To find the units digit of these numbers, we have some standard results, which we use as formula.

(Any even number)^{4n} = …6

It means that any even number raised to any power, which is a multiple of 4, will give 6 as the units digit.

(Any odd number)^{4n} = …1

It means that any odd number raised to any power, which is a multiple of 4, will give 1 as the units digit.

Exception: 0, 1, 5, 6 [These are independent of power, and units digit will be the same.]

Example 37 Find the units digit of $25{,}678^{2345} \times 3485^{4857}$.

Solution Units digit of $25{,}678^{2345}$ = units digit of 8^{45}

(To find the units digit, we need to have units digits only. Similarly, to find tens digit, we need to have the tens and units digit only. In the present case, we are considering only last two digits of the power because divisibility rule of 4 needs only the last two digits of the number.)

$$8^{45} = 8^{44+1} = 8^{44} \times 8^1 = (\ldots6) \times 8 = \ldots8$$

Example 38 What is the units digit of ${}_{32}32^{32}$?

Solution 32 is an even number that is having a power of the form 4^n. Therefore, it will give 6 as the units digit.

Example 39 When 3^{32} is divided by 50, it gives a number of the format (*asdf*…•*xy*) (*xy* being the last two digits after decimal). Find *y*.

Solution It can be observed that units digit of 3^{32} = 1. Now, any number having 1 as the units digit will always give 2 at the units place when divided by 50. Therefore, the answer is 2.

Example 40 What is the last non-zero digit of the number 30^{2720}?

Solution $30^{2720} = [30^4]^{680} = ...1$

Units digit can also be determined by cyclicity method. It can be seen that

Units digit of $2^1 = 2$
Units digit of $2^2 = 4$
Units digit of $2^3 = 8$
Units digit of $2^4 = 6$
Units digit of $2^5 = 2$

Therefore, it can be inferred that units digit of 2^1 = units digit of 2^5 = units digit of 2^9.

Hence, the cyclicity of 2 = 4, that is, every fourth power of 2 will give same units digit.

Similarly, cyclicity of 3 = 4

Cyclicity of 4 = 2
Cyclicity of 7 = 4
Cyclicity of 8 = 4
Cyclicity of 9 = 2

Cyclicity of 0 or Cyclicity of 1 or Cyclicity of 5 or Cyclicity of 6 = 1

Tens Digit

Method 1: Cyclicity Method

Digits	Cyclicity
2, 3, 8	20
4, 9	10
5	1
6	5
7	4

Example 41 What is the tens place digit of 12^{42}?

Solution For this, we need to break 12^{42} first by using binomial theorem as $(10 + 2)^{42}$. Obviously, this expression will have 43 terms, and out of these 43 terms, the first 41 terms will have both of their tens and units place digit as 0.

The last two terms will be ${}^{42}C_{41} \times 10^1 \times 2^{41} + {}^{42}C_{42} \times 10^0 \times 2^{42}$. Now, we will find the tens place digit of all these terms individually.

Tens digit of ${}^{42}C_{41} \times 10^1 \times 2^{41} = 42 \times 10 \times (02)$ [Cyclicity of 2 is 20; 2^{41} will have same tens digits as 2^1] = 840, and therefore, 40 are the last two digits.

Similarly, ${}^{42}C_{42} \times 10^0 \times 2^{42} = 1 \times 1 \times 04 = 04$. Finally, the last two digits are (40 + 04) = 44, and therefore, 4 is the tens place digit.

Note: $(25)^n$ and $(76)^n$ will always give 25 and 76 as the last two digits for any natural number value of n.

Method 2: Generalization Method

(i) (Any even number)^{20N} will give 76 as its last two digits (where N is any natural number).
However, if units digit = 0, then it will give '00' as the last two digits.

(ii) (Any odd number)^{20N} will give 01 as its last two digits (where N is any natural number). However, if units digit = 5, then it will give '25' as the last two digits.
Let us solve the previous worked-out example once again using this method.

Example 42 What is the tens place digit of 12^{42}?

Solution Using generalization (i),

we get $12^{20} = \ldots 76$ (76 as last two digits)

$12^{20} \times 12^{20} = 12^{40}$
$= (\ldots 76)$
$\times (\ldots 76)$
$= (\ldots 76)$

$12^{42} = 12^{40} \times 12^2 = (\ldots 76) \times (144)$

Since we are required to calculate the last two digits, we will focus only on the last two digits of both the numbers.

$(\ldots 76) \times (44) = 3344$. Hence, 44 is the last two digits of 12^{42}.

Note: we are not certain if 3 is at 100s place of this number.

Example 43 Find the tens place digit of 784^{1000}.

Solution Tens place digit of 784^{1000} = Tens place digit of $841,000$

As discussed earlier, (any even number)^{20N} will give 76 as the last two digits.

$84^{1000} = (84)^{20 \times 50} = (84)^{20N}$. This will have 76 as the last two digits.

Number of Exponents

Let us take a simple number 10^5

This is read as 10 to the power 5, or we can say that the exponent of 10 is 5.

In simple terms, exponents are also known as power.

Example 44 What is the maximum value of s if $N = (35 \times 45 \times 55 \times 60 \times 124 \times 75)$ is divisible by $5x$?

Solution If we factorize $N = (35 \times 45 \times 55 \times 60 \times 124 \times 75)$, then we can observe that 5 appears 6 times, it means N is divisible by 5^6.

Thus, the maximum value of $x = 6$

The exponent of any prime number P in $n!$ is given as

$$= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots\ldots \left[\frac{n}{p^x}\right],$$ where $n \geq p^x$ and [.] denotes the greatest integer value, that is, we have to consider only the integral value

Let us find the exponent of 5 in 1000! = $\frac{1000}{5}+\frac{1000}{5^2}+\frac{1000}{5^3}+\frac{1000}{5^4}=200+40+8+1=249$

Example 45 What is the highest power of 5 that can divide $N = (22! + 17{,}894!)$?

Solution The number of times this number is divisible by 5 is same as the number of zeroes at the end of this number. Here, 22! have 4 zeroes at its end, and therefore, N will also be having only four zeroes at its end. Hence, the highest power of 5 that can divide N is 4.

The process to find the exponent of any composite number in *n*! is given as

We have got three different kinds of composite numbers:

1. Product of two or more than two prime numbers with unit power of all the prime numbers.
   For example, $15(5 \times 3)$, $30(2 \times 3 \times 5)$, etc.
2. (Any prime number)^n = where $n > 1$. For example, $4(2^2)$, $27(3^3)$.
3. Product of two or more than two prime numbers with power of any one prime number more than 1.
   For example, $12(2^2 \times 3)$, $72(2^3 \times 3^2)$, etc.

Let us find the exponents of the abovementioned composite numbers:

1. Let us find the exponent of 15 in 100! 15 is the product of two distinct prime numbers 5 and 3. To find the exponents of 15, we need to find the exponents of 5 and 3 individually.
   Therefore, we will apply the same formula for finding the exponents for any prime number in both of these cases individually, and minimum of those two will be the solution.
   $100/5^x = [100/5] + [100/5^2] = 20 + 4 = 24$
   $100/3^x = [100/3] + [100/3^2] + [100/3^3] + [100/3^4]$
   $= 33 + 11 + 3 + 1 = 48$
   Obviously, 24 is the answer.
2. Let us find the exponent of 25 in 100!
   Similarly, we can find solution for the third category numbers also ($25 = 5^2$)
   In this case, we will first find the exponents of 5 and then divide it by 2 (actually the power) to find the exponents of 25.
   $100/5^x = [100/5] + [100/5^2] = 20 + 4 = 24$; $100/25^x = 24/2 = 12$
3. Similarly, we can find the solution for the third category numbers also.

Base System

In our decimal system of writing the numbers, we use 10 digits (0–9). In this system, the largest number of single digit is 9, and the moment we have to form a number larger than this number, we take the two-digit numbers starting from 10. Similarly, the largest number of two digits is 99, and after this, we have 100 (which is a three-digit number). Further, it is very plain and simple.

Now, let us assume a system of writing where we use only 6 digits (0–5). The largest single-digit number in this system will be 5 and next to this will be 10. Similarly, the largest two-digit number will be 55 and next is 100.

This whole procedure can be summed up in the following table:

$(0–9)_{10}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$(0–8)_9$	0	1	2	3	4	5	6	7	8	10	11	12	13	14	15	16	17	18
$(0–7)_8$	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17	20	21
$(0–6)_7$	0	1	2	3	4	5	6	10	11	12	13	14	15	16	20	21	22	23
$(0–5)_6$	0	1	2	3	4	5	10	11	12	13	14	15	20	21	22	23	24	25
$(0–3)_4$	0	1	2	3	10	11	12	13	20	21	22	23	30	31	32	33	100	101
$(0–2)_3$	0	1	2	10	11	12	20	21	22	100	101	102	110	111	112	120	121	122

Questions from this concept are asked in three different ways:

1. $(\text{Base})_{10}$ to any other base and vice versa
2. (Base) to (Base) and vice versa; none of x and y being equal to 10, but x and y will be given.
3. $(\text{Base})_x$ to $(\text{Base})_y$, the value of x and y will not be given.

   1. *(Base)$_{10}$ to any other base and vice versa*

   Method 1

   Let us consider (74).
   $(74)_{10} = 7 \times 10^1 + 4 \times 10^0$, since the base is 10.
   Now, if we have to convert this number to 9 base, then we will try to write it in terms of powers of 9.

   $(74)_{10} = 8 \times 9^1 + 2 \times 9^0 = (82)_9$
   $(74)_{10} = 1 \times 8^2 + 1 \times 8^1 + 2 \times 8^0 = (112)_8$
   $(74)_{10} = 1 \times 7^2 + 3 \times 7^1 + 4 \times 7^0 = (134)_7$
   $(74)_{10} = 2 \times 6^2 + 0 \times 6^1 + 2 \times 6^0 = (202)_6$

   While converting the numbers from decimal system to any other system of writing the numbers, we should be concerned with the following two rules:

- Take maximum possible power of the base and then keep writing rest of the number with the help of lesser power of base (as illustrated in the earlier example).
- Once we have used $(\text{base})^n$, where n is the maximum power, then we will be required to write the coefficient of all the powers of base from 0 to $(n - 1)$ as in the case of $(74)_{10} = (202)_6$.

  Now, let us assume that we have to convert $(356)_7$ in the base of 10. $(356)_7 = 3 \times 7^2 + 5 \times 7^1 + 6 \times 7^0 = (188)_{10}$

Method 2

Let us convert $(74)_{10}$ to the base of $(\;)_9$

Base	74	Reminder
9	8	2

Therefore, $(74)_{10} = (82)_9$

Let us convert $(74)_{10}$ to the base of $(\;)_8$

Base	74	Reminder
9	9	2
8	1	1

Therefore, $(74)_{10} = (112)_8$.

Let us convert $(74)_{10}$ to the base of $(\;)_7$

Base	74	Reminder
7	10	4
7	1	3
	1	

Therefore, $(74)_{10} = (134)_7$.

Let us convert $(74)_{10}$ to the base of $(\;)_6$

Base	74	Reminder
6	12	2
6	2	0
	Quotient	

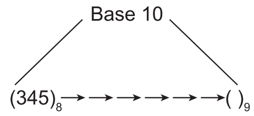
Therefore, $(74)_{10} = (202)_6$.

Task for students

Convert $(123)_{10}$ into base 9, base 8, base 7, base 15, base 20. Answers are given at the end of topic.

$(\text{Base})_x$ to $(\text{Base})_y$ and vice versa; none of x and y being equal to 10 but x and y will be given.

Converting $(345)_8$ to the base of $(\;)_9$:



We will do this problem with the help of creating a bridge of base 10 between base 8 and base 7.

Step 1 Convert $(345)_8$ to base 10.

$345 = 3 \times 8^2 + 4 \times 8^1 + 5 \times 8^0 = (229)_{10}$

Step 2 Now, convert this number in base 10 to base 9.

$(229)_{10} = 2 \times 9^2 + 7 \times 9^1 + 4 \times 9^0 = (274)_9$

However, if new base is a power of old base and vice versa, then it can be converted directly to the new base, that is, it is not necessary to go to base 10 for these types of conversions.

Let us convert (74) to the base of $(\;)_8$:

Converting $(101110010)_2$ to Octal $(\;)_8$ system:

At first, we will club three digits of binary number into a single block, and then, we will write the decimal equivalent of each group (left to right).

Therefore, $(101110010)_2$ is now $(101)_2(110)_2(010)_2$.

Now, $(101)_2 = 1 \times 2^2 + 0 + 1 \times 2^0 = 5$

$(110)_2 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 6$

$(010)_2 = 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 2$.

Thus, $(101110010)_2 = (562)_8$

Converting $(101110010)_2$ to hexa-decimal $(\;)_{16}$ system:

At first, we will club four digits of binary number into a single block, and then, we will write the decimal equivalent of each group (left to right).

Therefore, $(101110010)_2$ is now $(0001)_2(0111)_2(0010)_2$.

Now, we have the following:

Decimal equivalent of $(0001)_2 = 1$

Decimal equivalent of $(0111)_2 = 7$

Decimal equivalent of $(0010)_2 = 2$

$(101110010)_2 = (172)_{16}$

$(\text{Base})_x$ to $(\text{Base})_y$, value of x and y will not be given.

Normally, in these types of questions, some calculation is given in some unknown system of writing numbers. On the basis of that calculation, we will be required to solve questions.

Example 46 In a system of writing of N digits, we have $4 \times 6 = 30$ and $5 \times 6 = 36$. What will be the value of $N = 3 \times 4 \times 5$ in the same system of writing?

Solution Let us assume that there are N digits in this system of writing.

$$(30)_N = 3 \times N + 0 \times N^0 = 24$$
$$\Rightarrow \quad 3N = 24$$
$$\Rightarrow \quad N = 8$$

Therefore, this system of writing has 8 digits.

In this system, $3 \times 4 \times 5 = 60$ will be written as 74 ($60 = 7 \times 8^1 + 4 \times 8^0$).

Alternatively, since this system is having 6 as one of its digits, minimum value of N will be 7. Again, 24 is written as 30 in this system, then N is less than 10. Now, use trial-and error method for $N = 7$ or 8 or 9 to find N in $24 = (30)_N$.

Decimal Calculation

So far, we have seen the calculations involving natural numbers only. Now, let us work with decimals.

Let us see the process of converting decimal system numbers to any other system:

Let us assume that (12.725) is a number in decimal system, which is required to be converted into octal system (8 digits).

We will first convert 12 into octal system, i.e., $(12)_{10} = (14)_8$

Now, to convert (0.725) into $(\ )_8$, we will apply the following method:

$0.725 \times 8 = \mathbf{5}.8$ eliminate the integral part from here.

$0.8 \times 8 = \mathbf{6}.4$ eliminate the integral part from here.

$0.4 \times 8 = \mathbf{3}.2$ eliminate the integral part from here.

$0.2 \times 8 = \mathbf{1}.6$ eliminate the integral part from here.

Further, we keep doing this until we get decimal part as zero, that is, the product should be an integer.

$(0.725)10 = (0.5632\ldots)_8$

Thus, $(12.725) = (14.5632\ldots)_8$

Let us discuss the process of converting any other system numbers to decimal system:

Now, let us assume that if $(15.453)_7$ is to be converted into decimal system, then the process is as follows:

We will first convert $(15)_7$ into decimal system.

$(15)_7 = 1 \times 7^1 + 5 \times 7^0 = (12)_{10}$

Let us discuss the basic algebraic calculations involving the base systems:

Addition

$$\begin{array}{r} 325_7 \\ +456_7 \\ \hline \end{array}$$

Start with the units place digit, 5 + 6 = 11, which is 14_7. Thus, units digit is 4 and carry over is 1.

Next is tens place digit, 2 + 5 + 1 (carry over) = 8, which is 11_7. Hence, tens digit is 1 and carry over is again 1.

Next is 3 + 4 + 1 (carry over) = 8, which is 11_7.

$$\begin{array}{r} 325_7 \\ +456_7 \\ \hline 1114_7 \\ \hline \end{array}$$

Subtraction

$$456_8 - 367_8$$

Let us start with the units digit; since 6 is smaller than 7, we will borrow 1 from the tens place digit. Therefore, now, it is 14 (when the base is 10, we get 10; however, here, the base is 8, and hence, we will get 8.). When 7 subtracted from 14, we have 7, which is the units digit.

Next, tens digit is now 4 and we have to subtract 5 from it. We will again borrow 1 from hundreds place digit. Now, it is 12, and 12 – 6 = 6, which is the tens place digit.

Now, hundreds place digit is 3(4 – 1), and (3 – 3) = 0.

$$\begin{array}{r} 456_8 \\ -367_8 \\ \hline 67_8 \\ \hline \end{array}$$

Note: Another method of calculation is (i) converting these values (in whatever base) into decimal system, (ii) performing the actual calculation in decimal system itself, and (iii) converting the numbers into the required or given system.

The following are some of the standard system of writing:

Decimal system
Digits used—0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
Total digits used = 10 digits
Hexa-decimal system
Digits used—0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. Total digits used = 16
Octal system
Digits used—0, 1, 2, 3, 4, 5, 6, 7.
Total digits used = 8
Binary system
Digits used—0, 1
Total digits used—2

The divisibility rules for systems other than decimal system are as follows:

Here, we emphasize that different number systems are just different ways to write numbers. Thus, the divisibility of one number by another does not depend on the particular system in which they are written.

At the same time, in each system, there are some techniques to determine divisibility by certain specific numbers. These are the divisibility tests.

Now, let us investigate the other, less trivial divisibility tests. Perhaps, the most well-known of these are the tests for divisibility by 3 and 9. We will try to generalize these tests for any number base system. Is $1{,}23{,}45{,}65{,}64{,}231_7$ divisible by 6?

We know the divisibility rule for 9—sum of digits of the number should be divisible by 9.

Sum of digits of this number is 42.

Now, we can answer this question easily: since the sum of the digits (which is 42_{10}) is divisible by 6, the number itself is divisible by 6.

In general, the sum of the digits of a number written in the base *n* system is divisible by (*n* – 1) if and only if the number itself is divisible by (*n* – 1).

Therefore, divisibility rule for 4 in a base system of 5—sum of digits of the number should be divisible by 4. For example, 31_5 is divisible by 4.

Similarly, if we have to find the divisibility rule of 12 in the base of 11, it will be nothing but same as the divisibility rule of 11 in the base of 10. Generalizing the whole concept, we can say that the divisibility rule of any natural number N in the base of $(N - 1)$ will be same as divisibility rule of 11 on base 10.

Pigeonhole Principle

Despite not being very much in vogue with respect to the CAT preparation (only a few questions have been asked from this concept so far in CAT), the importance of this topic lies in the fact that this concept is purely logical.

General Statement of Pigeonhole Principle

If we assume $(N + 1)$ or more pigeons in N holes (nests), then at least one hole will be there, which will have 2 or more pigeons.

Example 47 What is the minimum number of people in any group of five people who have an identical number of friends within the group, provided if A is friend of B, then B is also friend of A?

Solution Since there are five persons in the group, the possible number of friends is 0, 1, 2, 3, 4. It seems here that everybody is having different number of friends, and hence, the answer is zero. However, anybody having four friends ensures that nobody is having 0 friends. Thus, at least two persons must have same number of friends.

Practice Exercises

Warm Up

Q.1 Which of the following is the smallest?
(a) $5^{1/2}$ (b) $6^{1/3}$ (c) $8^{1/4}$ (d) $12^{1/6}$

Q.2 A number N is divisible by 6 but not divisible by 4. Which of the following will not be an integer?
(a) $N/3$ (b) $N/2$ (c) $N/6$ (d) $N/12$

Q.3 If a, b, and c are consecutive positive integers, then the largest number that always divides $(a^2 + b^2 + c^2)$
(a) 14 (b) 55
(c) 3 (d) None of these

Q.4 $\dfrac{(3.134)^3 + (1.866)^3}{(3.134)^2 - 3.134 \times 1.866 + (1.866)^2} = ?$
(a) 25 (b) 2.68 (c) 1.038 (d) 5

Q.5 If n^2 is a perfect cube, then which of the following statements is always true?
(a) n is odd.
(b) n is even.
(c) n^3 is a perfect square.
(d) n is a perfect cube.

Q.6 If $(5x + 11y)$ is a prime number for natural number values of x and y, then what is the minimum value of $(x + y)$?
(a) 2 (b) 3 (c) 4 (d) 5

Q.7 For what values of x is $(25^x + 1)$ divisible by 13?
(a) All real values of x
(b) Odd natural values of x
(c) Even values of x
(d) All the integral values of x

Q.8 Which of the following numbers lies between 5/6 and 6/7?
(a) 71/84 (b) 31/42
(c) 129/168 (d) 157/339

Q.9 By multiplying with which of the following numbers, does the product of $8 \times 9 \times 10 \times 11 \times 12$ become a perfect square?
(a) 55 (b) 11 (c) 165 (d) 310

Q.10 What is the difference between the sum of the cubes and that of squares of the first 10 natural numbers?
(a) 5280 (b) 2640 (c) 3820 (d) 4130

Q.11 If $3 - 9 + 15 - 21 + ...$ up to 19 terms $= x$ then x is a/an
(a) odd number (b) even number
(c) prime number (d) irrational number

Q.12 What is the units digit of $21^3 \times 21^2 \times 34^7 \times 46^8 \times 77^8$?
(a) 4 (b) 8 (c) 6 (d) 2

Q.13 If the units digit in the product $(47n \times 729 \times 345 \times 343)$ is 5, what is the maximum number of values that n may take?
(a) 9 (b) 3 (c) 7 (d) 5

Q.14 In how many ways, can 846 be resolved into two factors?
(a) 9 (b) 11
(c) 6 (d) None of these

Q.15 If a number is divided by 15, it leaves a remainder of 7. If thrice the number is divided by 5, then what is the remainder?
(a) 5 (b) 6 (c) 7 (d) 1

Q.16 A number when divided by 391 gives a remainder of 49. Find the remainder when it is divided by 39.
(a) 10 (b) 9
(c) 11 (d) Cannot be determined

Q.17 p and q are two prime numbers such that $p < q < 50$. In how many cases, would $(q + p)$ be also a prime number?
(a) 5 (b) 6
(c) 7 (d) None of these

Q.18 How many distinct factors of 1600 are perfect cubes?
(a) 3 (b) 4 (c) 6 (d) 2

Q.19 The LCM of 96,144 and N is 576. If their HCF is 48, then which of the following can be one of the values of N?
(a) 168 (b) 192 (c) 144 (d) 244

Q.20 If p and q are consecutive natural numbers (in increasing order), then which of the following is true?
(a) $q^2 < p$ (b) $2p > p^2$
(c) $(q + 1)^2 > p^2$ (d) $(p + 2)^3 < q^3$

Q.21 $(17^{21} + 19^{21})$ is not divisible by
(a) 36 (b) 8 (c) 9 (d) 18

Q.22 Which of the following will divide $11^{12296} - 1$?
(a) 11 and 12 (b) 11 and 10
(c) 10 and 12 (d) 11 only

Q.23. If a, b, c, and d are consecutive odd numbers, then $(a^2 + b^2 + c^2 + d^2)$ is always divisible by
(a) 5 (b) 7 (c) 3 (d) 4

Q.24. Four bells toll at intervals of 14, 21, and 42 min, respectively. If they toll together at 11:22 am, when will they toll together for the first time after that?
(a) 11:56 am (b) 12:04 pm
(c) 12:06 pm (d) 11:48 am

Q.25 When x is divided by 6, remainder obtained is 3. Find the remainder when $(x^4 + x^3 + x^2 + x + 1)$ is divided by 6.
(a) 3 (b) 4 (c) 1 (d) 5

Q.26 I have 7^7 sweets and I want to distribute them equally among 2^4 students. After each of the student got maximum integral sweets, how many sweets are left with me?
(a) 8 (b) 5
(c) 1 (d) None of these

Q.27 When I distribute some chocolates to my 40 students, three chocolates will be left. If I distribute the same number of chocolates to my students and my colleague Manoj Dawrani, seven chocolates are left. Find the minimum number of chocolates I have.
(a) 1443 (b) 1476
(c) 1480 (d) None of these

Q.28 The LCM of two numbers is 40 times of their HCF. The sum of the LCM and HCF is 1476. If one of the numbers is 288, find the other numbers.
(a) 169 (b) 180 (c) 240 (d) 260

Q.29 1010101...94 digits is a 94-digit number. What will be the remainder obtained when this number is divided by 375?
(a) 10 (b) 320
(c) 260 (d) None of these

Q.30 Chandrabhal adds first N natural numbers and finds the sum to be 1850. However, actually one number was added twice by mistake. Find the difference between N and that number.
(a) 40 (b) 33 (c) 60 (d) 17

Q.31 When I distribute a packet of chocolates to 7 students, I am left with 4 chocolates. When I distribute the same packet of chocolates to 11 students, I am left with 6 chocolates. How many chocolates will be left with me if I distribute the same packet of chocolates among 13 students (a packet of chocolate contains total number of N chocolates, where $1000 < N < 1050$)?
(a) 2 (b) 0 (c) 6 (d) 7

Q.32 How many prime numbers are there between 80 and 105?
(a) 3 (b) 4 (c) 5 (d) 8

Q.33 If x and y are consecutive natural numbers in an increasing order, then which of the following is always true?
(a) $x^y > y^x$ (b) $y^x > x^y$
(c) $x^x > y^y$ (d) $y^y > x^x$

Q.34 What is the remainder when 5^{79} is divided by 7?
(a) 1 (b) 0 (c) 5 (d) 4

Foundation

Q.1 The LCM of two natural numbers is 590 and their HCF is 59. How many sets of values are possible?
(a) 1 (b) 2
(c) 5 (d) 10

Q.2 MUL has a waiting list of 5005 applicants. The list shows that there are at least 5 males between any two females. The largest number of females in the list could be:
(a) 920 (b) 835 (c) 721 (d) 1005

Q.3 HCF of two numbers A and B is 24. HCF of two other numbers C and D is 36. What will be the HCF of A, B, C, and D?
(a) 12 (b) 24 (c) 36 (d) 6

Q.4 How many zeroes will be there at the end of $25 \times 35 \times 40 \times 50 \times 60 \times 65$?
(a) 6 (b) 8 (c) 5 (d) 7

Q.5 What is the units digit of $576{,}847 \times 564{,}068 \times 96{,}467 \times 458{,}576$?
(a) 2 (b) 4 (c) 6 (d) 8

Q.6 What is the units digit of $1! + 2! + 3! + 99! + 100!$?
(a) 3 (b) 1 (c) 5 (d) 6

Q.7 How many divisors will be there of the number 1020?
(a) 12 (b) 20 (c) 24 (d) 36

Q.8 In Q.7, what is the difference between the number of even divisors and number of prime divisors?
(a) 13 (b) 12
(c) 11 (d) None of these

Q.9 $N = 7!^3$. How many factors of N are multiples of 10?
(a) 736 (b) 1008
(c) 1352 (d) 894

Q.10 A number N has odd number of divisors. Which of the following is true about N?
(a) All the divisors of this number will be odd.
(b) There will be at least $(N - 11)$ prime divisors.
(c) N will be a perfect square.
(d) At least one divisor of the number should be odd.

Q.11 How many zeroes will be there at the end of the expression $N = 2 \times 4 \times 6 \times 8 \times \ldots \times 100$?
(a) 10 (b) 12
(c) 14 (d) None of these

Q.12. How many zeroes will be there at the end of the expression $N = 10 \times 20 \times 30 \ldots \times 1000$?
(a) 1280 (b) 1300
(c) 1320 (d) None of these

Q.13 How many zeroes will be there at the end of the expression $N = 7 \times 14 \times 21 \times \ldots \times 777$?
(a) 24 (b) 25
(c) 26 (d) None of these

Q.14 The number from 1 to 33 are written side by side as follows: 123,456... 33. What is the remainder when this number is divided by 9?
(a) 0 (b) 1 (c) 3 (d) 6

Q.15 The number 444,444 ... (999 times) is definitely divisible by:
(a) 22 (b) 44
(c) 222 (d) All of these

Q.16 Find the units digit of $N = {}_{17}27!^{37!}$
(a) 1 (b) 3 (c) 7 (d) 9

Q.17 How many divisors of $N = 420$ will be of the form $(4n + 1)$, where n is a whole number?
(a) 3 (b) 4 (c) 5 (d) 8

Q.18 $N = 2^3 \times 5^3 \times 7^2$. How many sets of two factors of N are co-prime?
(a) 72 (b) 64
(c) 36 (d) None of these

Q.19 What is the units digit of $2^{3^{4^5}}$?
(a) 2 (b) 4 (c) 8 (d) 6

Q.20 How many zeroes will be there at the end of $1003 \times 1001 \times 999 \times \ldots \times 123$?
(a) 224 (b) 217
(c) 0 (d) None of these

Q.21 How many zeroes will be there at the end of $36!^{36!}$?
(a) $7 \times 6!$ (b) $8 \times 6!$
(c) $7 \times 36!$ (d) $8 \times 36!$

Q.22 The number formed by writing any digit 6 times (e.g., 111,111, 444,444, etc.) is always divisible by:
(i) 7 (ii) 11 (iii) 13
(a) (i) and (ii) (b) (ii) and (iii)
(c) (i) and (iii) (d) (i), (ii) and (iii)

Q.23 What is the maximum value of HCF of $[n^2 + 17]$ and $(n + 1)^2 + 17]$?
(a) 69 (b) 85
(c) 170 (d) None of these

Q.24 What is the number of pairs of values of (x, y), which will satisfy $2x - 5y = 1$, where $x < 200$, and x and y are positive integers?
(a) 38 (b) 39 (c) 40 (d) 41

Q.25 $N = 2^3 \times 5^3$. How many sets of two distinct factors of N are co-prime to each other?
(a) 12 (b) 24 (c) 23 (d) 11

Q.26 What is the sum of digits of the least multiple of 13, which when divided by 6, 8, and 12 leave 5, 7, and 11 as the remainder?
(a) 5 (b) 6 (c) 7 (d) 8

Q.27 What is the units digit of $7^{11^{22^{33}}}$?
(a) 1 (b) 3 (c) 7 (d) 9

Q.28 What is the remainder when $(1! + 2! + 3! + \ldots 1000!)$ is divided by 5?
(a) 1 (b) 2 (c) 3 (d) 4

Q.29 If $A = 3^{150} \times 5^{76} \times 7^{140}$, $B = 3^{148} \times 5^{76} \times 7^{141}$, $C = 3^{148} \times 5^{80} \times 7^{139}$, and $D = 3^{151} \times 5^{80} \times 7^{142}$, then the order of A, B, C, and D from largest to smallest is:
(a) DACB (b) CDBA
(c) CDAB (d) DCAB

Q.30 The HCF of 0.3, 0.15, 0.225, 0.0003 is:
(a) 0.0003 (b) 0.3 (c) 0.15 (d) 0.0015

Q.31 How many numbers between 1 and 250 are divisible by 5 but not by 9?
(a) 98 (b) 97
(c) 101 (d) None of these

Q.32 A and B are two distinct digits. If the sum of the two-digit numbers formed by using both the digits is a perfect square, what is the value of $(A + B)$?
(a) 9 (b) 11 (c) 13 (d) 17

Q.33 A number $N = 897324P64Q$ is divisible by both 8 and 9. Which of the following is the value of $(P + Q)$?
i. 2 ii. 11 iii. 9
(a) Either i or ii (b) Either ii or iii
(c) Either i or ii or iii (d) None of these

Direction for Questions 34 and 35: Read the following passage below and solve the questions based on it.

A = Set of first N positive numbers. There are 16 numbers in A that are divisible by both X and Y. There are 50 numbers in A that are divisible by X but not by Y and 34 numbers in A divisible Y but not by X.

Q.34 How many numbers in A are divisible by any of the two numbers?
(a) 100 (b) 50
(c) 200 (d) None of these

Q.35 How many numbers in N are divisible by X?
(a) 42 (b) 56
(c) 66 (d) None of these

Q.36 Nitin had forgotten his 6-digit bank account number but only remembered that it was of the form $X515X0$ and was divisible by 36. What was the value of X?
(a) 4 (b) 7 (c) 8 (d) 9

Q.37 Students from the Delhi Public School are writing their exams in Kendriya Vidyalaya. There are 60 students writing their Hindi exams, 72 students writing French exam, and 96 students writing their English exam. The authorities of the Kendriya Vidyalaya have to make arrangements such that each classroom contains equal number of students. What is the minimum number of classrooms required to accommodate all students of Delhi Pubic School?
(a) 19 (b) 38 (c) 13 (d) 6

Q.38 In the Jyotirmayi school, all classes started at 9:00 am. The school has three sections: primary, middle, and secondary. Each class for the primary section lasts for 30 min, for the middle section for 45 min, and for the secondary section for 30 min. A lunch break has to be given for the entire school when each of three sections have just finished a respective class and are free. What is the earliest time for the lunch break?
(a) 11:00 am (b) 10:30 am
(c) 12:00 pm (d) 12:30 pm

Q.39 In the firing range, four shooters are firing at their respective targets. The first, the second, the third, and the fourth shooter hit the target once every 5 s, 6 s, 7 s, and 8 s, respectively. If all of them hit their target at 10:00 am, when will they hit their target together again?
(a) 10:14 am (b) 10:28 am
(c) 10:30 am (d) 10:31 am

Q.40 Two friends Harry and Jayesh were discussing about two numbers. They found the two numbers to be such that one was twice the other. However, both had the same number of prime factors, while the larger one had 4 more factors than the smaller one. What are the numbers?
(a) 40, 80 (b) 20, 40
(c) 30, 60 (d) 50, 100

Q.41 To celebrate their victory in the World Cup, the Sri Lankans distributed sweets. If the sweets were distributed among 11 players, 2 sweets were left. When the sweets were distributed to 11 players, 3 extra players, and 1 coach, even then 2 sweets were left. What is the minimum number of sweets in the box?
(a) 167 (b) 334 (c) 332 (d) 165

Q.42 The first 20 natural numbers from 1 to 20 are written next to each other to form a 31-digit number N = 12345 67891011121314151617181920. What is the remainder when this number is divided by16?
(a) 0 (b) 4 (c) 7 (d) 9

Q.43 Two friends Kanti and Sridhar were trying to find the HCF of 50 distinct numbers. If they were finding the HCF of two numbers at a time, how many times this operation should be repeated to find the HCF of 50 numbers?
(a) 20 (b) 25 (c) 49 (d) 50

Q.44 How many zeroes will be there at the end of $N = 18! + 19!$?
(a) 3 (b) 4
(c) 5 (d) Cannot be determined

Q.45 Manish was dividing two numbers by a certain divisor and obtained remainders as 437 and 298, respectively. When he divides the sum of the two numbers by the same divisor, the remainder is 236. What is the divisor?
(a) 499 (b) 735
(c) 971 (d) None of these

Q.46 I purchased a ticket for the football match between France and Italy in the World Cup. The number on the ticket was a 5-digit perfect square such that the first and