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Authentic

SHORTCUTS, TIPS, TRICKS & TECHNIQUES in MATHEMATICS for JEE MAIN, ADVANCED & KVPY

Er. Vaibhav Singh

Strategic Book for
Class 11/ 12 & Engineering Exams

EBD_7325

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FOREWORD

The competitive exams like JEE test an aspirant's conceptual knowledge & how fast he/ she solve the problems with accuracy. So it becomes necessary that the students should know the short-cut methods in addition to the traditional methods of analysis. Keeping this in mind DISHA Publication brings a unique & innovative book *Authentic* **SHORTCUTS, TIPS, TRICKS & TECHNIQUES in MATHEMATICS** for JEE Main, Advanced & KVPY to enable aspirants for advanced abilities to Solve KVPY, JEE Main & Advanced level Questions well within the stipulated time.

An earnest effort has been made to bring the book *Authentic* **SHORTCUTS, TIPS, TRICKS & TECHNIQUES in MATHEMATICS**. We have really worked hard researching for the best possible Tips, Tricks, Techniques and Shortcut Solutions which students must know and can utilize in the examination hall.

- Shortcuts to help you in providing a different perspective to a concept/ problem thus strengthening your conceptual understanding.
- Tips provide you the Most Important Points to remember that aids in Conceptual Understanding & Problem Solving.
- Tricks empower you with magical tools that help you develop unique approaches to solve a problem.
- Shortcut Solutions provides alternate faster methods that save you a lot of time during examination.

The book encompasses 26 Chapters, which start with Review of Key Notes and Formulae, followed by Shortcuts, Tips, Tricks and Techniques which are further followed by Illustrations demonstrating Shortcut Solutions. The book in all contains:

1. 250+ Chapter-wise Shortcuts, Tips & Tricks to solve JEE Level Problems.
2. 400+ Illustrations with Shortcut Solutions of JEE Level Questions including JEE Past Years Questions.
3. 25+ International Techniques to crack JEE Advanced Level Questions.
4. 500+ Chapter-wise JEE Level Questions Exercise with Accurate & Shortest Possible Solutions.
5. Chapter-wise 350+ Important Key Notes Formulae.

This book provides you with hundreds of short-cut methods for the most conceptual and relevant problems. This book can also be used as a REVISION BOOK for various competitive exams. I hope that the book will fulfill the needs of the students for which it has been designed. We have made our best efforts to keep the book error-free but some errors might have crept in by mistake. We request our readers to highlight these errors and their fruitful suggestions so that we can keep on improving this book.

Author: Er. Vaibhav Singh

No Matter where You Prepare from, keep this book as your companion.
It would definitely improve your score by 25-30%.

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Contents

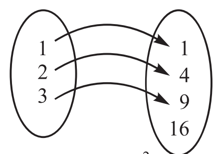
UNIT-I : CALCULUS

1 Functions

Review of Key Notes and Formulae

Definition: If A and B are two non-empty sets, then the rule that, for each and every element of set A is uniquely associate with set B.

$f : A \rightarrow B$



$f(x) = x^2$

Domain: All elements of set A

$$D_f = \{1, 2, 3\}$$

Co-domain: All elements of Set B

$$Co\text{-}D_f = \{1, 4, 9, 16\}$$

Range: Elements of set B which are involved in mapping.

$$R_f = \{1, 4, 9\}$$

Different Types of Functions

1. **Polynomial function:** Function in the form of:

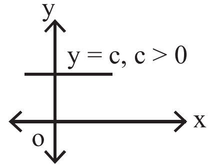
$$f(x) = a_0 x^n + a_1 x^{n-1} + \text{..........} + a_n \; ; a_0 \neq 0 \; ; \text{Degree} = n$$

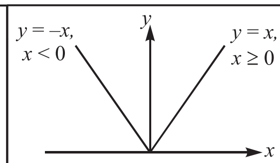
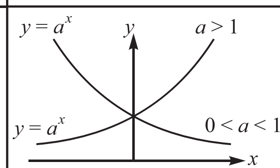
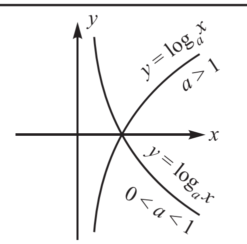
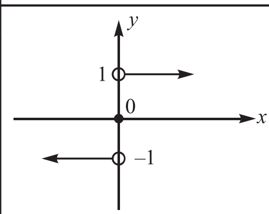
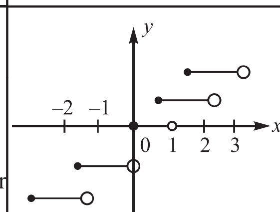
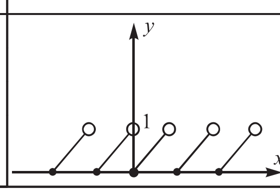
where, $n, n-1, n-2$ .... are non-negative integers. Domain of $f(x) = R$

2. **Rational function:** Functions in form of

$$f(x) = \frac{p(x)}{q(x)} \; ; q(x) \neq 0$$

where, $P(x)$ and $q(x)$ are polynomial in x. Domain of $f(x) = R - \{x : q(x) = 0\}$

	Function	Graph	Domain & Range
3.	**Constant function:** y = f(x) = c $\quad \forall x \in R$, where c is a constant	 	Dom : $x \in R$ Range : $y = \{c\}$

4.	**Modulus function:** $y = f(x) = \lvert x \rvert$		Dom: $x \in R$ Range: $y \in [0, \infty]$
5.	**Exponential function:** $y = f(x) = a^x,$ where $a > 0, a \neq 1$		Dom: $x \in R$ Range: $y \in (0, \infty)$
6.	**Logarithmic function:** $y = f(x) = \log_a x$ where $a > 0, a \neq 1$		Dom: $x \in (0, \infty)$ Range: $y \in R$
7.	**Signum function:** $y = f(x) = \text{Sgn}(x)$ $\Rightarrow \ f(x) = \begin{cases} \frac{\lvert x \rvert}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$		Dom: $x \in R$ Range: $y \in \{-1, 0, 1\}$
8.	**Greatest integer function:** $y = f(x) = [x]$ x if x is integer; greatest integer less than x		Dom: $x \in R$ Range = $\{z\}$
9.	**Fractional part function:** $y = f(x) = \{x\}$ $\{x\} = x - [x]$		Dom: $x \in R$ Range: $y \in [0, 1)$
10.	**Trigonometric function:**		

Functions	Domain	Range
$y = \sin x$	$x \in R$	$y \in [-1, 1]$
$y = \cos x$	$x \in R$	$y \in [-1, 1]$

$y = \tan x$	$x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$	$y \in R$
$y = \cot x$	$x \in R - \{n\pi\}$	$y \in R$
$y = \text{cosec } x$	$x \in R - \{n\pi\}$	$y \in (-\infty, -1] \cup [1, \infty)$
$y = \sec x$	$x \in R - \{(2n+1)\frac{\pi}{2}\}$	$y \in (-\infty, -1] \cup [1, \infty)$

Equal or Identical Functions:

Two functions $f(x)$ and $g(x)$ are said to be identical if.

(i) Domain of $f(x)$ = Domain of $g(x)$

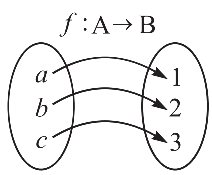
(ii) Co-domain of $f(x)$ = Co-domain of $g(x)$

(iii) $f(x) = g(x)$ for every x belonging to their domain.

Classification of Functions:

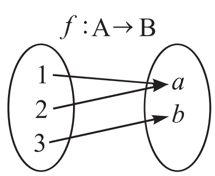
1. **One-one function:** The mapping $f : A \to B$

   (Injective Function) is one-one function if different elements in A have different images in B.



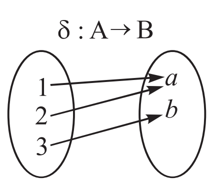
$$\boxed{\begin{array}{l} x_1, x_2 \in A \\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \end{array}}$$

2. **Many-one function:** The mapping $f : A \to B$ is many-one two or more than two different elements in A have the same image in B.



$$\boxed{\begin{array}{l} x_1, x_2 \in A \\ x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2) \end{array}}$$

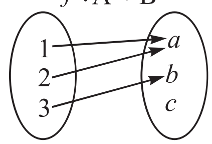
3. **Onto (surjective) function:** A function is said to be onto if



$$\boxed{\text{Range} = \text{Co-domain}}$$

4. **Into function:** A function is said to be into if $\boxed{\text{Range} \neq \text{Co-domain}}$

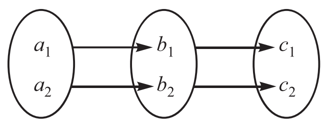
$f : \text{A} \rightarrow \text{B}$



★**Note** $\Rightarrow$ $\boxed{\text{Bijective function} \Leftrightarrow \text{One-one} + \text{Onto}}$

Composition of Function:

If $f : A \rightarrow B$ and $g : B \rightarrow C$ are two functions, then composite function of f and g are $go'f : A \rightarrow C$ will be defined as $g\,(f\,(x)) = gof\,(x)\ \forall\ x \in A$



Even and Odd Functions

(i) If $f\,(x) + f\,(-x) = 0\ \forall\ x \in$ Domain of $f\,(x)$

$\Rightarrow$ Odd function $\Rightarrow$ Symmetric about origin

(ii) If $f\,(x) = f\,(-x)\ \forall\ x \in$ Domain of $f\,(x)$

$\Rightarrow$ Even function $\Rightarrow$ Symmetric about y-axis

★**Note** $f\,(x) = 0$ is even as well as odd function.

Homogeneous Function:

Functions consists of variables both x & y such that $f\,(x, y)$ is homogeneous if:

$$f\,(tx, ty) = t^n f\,(x, y)$$

$$\downarrow$$

Homogeneous of degree 'n'

Periodic Function:

A function is periodic if its each value is repeated after a definite interval. So a function is periodic if there exists a positive real number 'T' such that

$$f\,(x + T) = f\,(x)\ \forall\ x \in D_f$$

★ Period $= nT\,;\ n \in I$

$\therefore$ $\boxed{f\,(x + nt) = f\,(x)}\ ;\ n \in I$

★**Note:** Constant function has no fundamental period.

Inverse of a Function:

If $f : A \to B$ is a one-one and onto fn. both then we can define the inverse of the function as $g : B \to A$, such that $f(x)\ y \Rightarrow \underset{\text{Inverse of } f(x).}{\underset{\downarrow}{g(y)}} = x$

Properties of Invertible Function:

(i) Inverse of bijective function is unique and bijective.

(ii) $(f^{-1})^{-1} = f$

(iii) $(gof)^{-1} = f^{-1}og^{-1}$

(iv) Inverse of a function is a mirror image about $y = x$ line.

TIPS AND TRICKS: (T-1)

There are only two polynomial functions exists, which satisfies the condition $f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right)$

are $\boxed{f(x) = 1 \pm x^n}$; $n \in I^+ : x \in R$

Illustration 1

Find the polynomial function which satisfies the condition $f(x) + \left(\frac{1}{x}\right) = f(x).\left(\frac{1}{x}\right)$ of degree 3 and is always increasing function.

Short-cut solution :

$\boxed{\text{Using T-1}}$ $f(x) = 1 \pm x^3$ ($\because$ degree is 3)

Now, for function is always $\uparrow$sing

So, $f(x) = 1 + x^3$ or $f(x) = 1 - x^3$

Differentiate, $f'(x) = 3x^2 > 0$ & $f'(x) = -3x^2 < 0$

$\Rightarrow$ $\uparrow$sing $\Rightarrow$ $\uparrow$sing fn

Hence, we conclude that the required function is

$f(x) = 1 + x^3$

TIPS AND TRICKS: (T-2)

If $y = f(x) = [x]$ is a greatest integer function then,

$$[x] + [-x] = \begin{cases} 0 \ ; & x \in I \\ -1 ; & x \notin I \end{cases}$$

Illustration 2

Find the value of $\int_a^{3\pi}[2\cos x]dx$, where [ ] is the greatest integer function.

Short-cut solution :

$\boxed{\text{Using T-2}}$ $\int_a^{3\pi}[2\cos x]\,dx$, where [ ] is greatest integer function.

Let $I = \int_a^{3\pi}[2\cos x]\,dx$ ... (1)

$\boxed{\text{Apply, King Property : } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx}$

$\Rightarrow$ $I = \int_0^{3\pi}[-2\cos x]\,dx$ ... (2)

Now, add (1) and (2)

$\Rightarrow$ $2I = \int_0^{3\pi}[2\cos x]+[-2\cos x]\,dx$

Since 2 cos x is not always '0' or Integer

$\Rightarrow$ $2I = \int_0^{3\pi}(-1)\,dx$

$\therefore I = \dfrac{-3\pi}{2}$

TIPS AND TRICKS: (T-3)

If $y = f(x) = \{x\}$ is a fractional part function.

then, $\{x\} + \{-x\} = \begin{cases} 0 \;; & x \in I \\ 1 \;; & x \notin I \end{cases}$

TIPS AND TRICKS: (T-4)

${}^nC_r$ and ${}^rC_n$, simultaneously possible only when $n = r$

Illustration 3

Let $f(x) = {}^{x+1}C_{2x-8}$, $g(x) = {}^{2x-8}C_{x+1}$ if $h(x) = f(x) \cdot g(x)$.

Then find domain and range of $h(x)$.

Short-cut solution :

$\boxed{\text{Using T-4}}$ $x + 1 = 2x - 8$

$\Rightarrow$ $x = 9$

Hence domain is $x \in \{9\}$ and $R_f = 1$.

TIPS AND TRICKS: (T-5)

Range of the function $f(x) = \dfrac{ax+b}{cx+d}$; $x \neq \dfrac{-d}{c}$ is $R - \left(\dfrac{d}{c}\right)$

Illustration 4

Find the range of the function $f(x) = \dfrac{2x+1}{5x-2}$; $x \neq \dfrac{2}{5}$

Short-cut solution :

$\boxed{\text{Using T-5}}$ $y \in R - \left\{\dfrac{2}{5}\right\}$

TIPS AND TRICKS: (T-6)

To find Range of $f(x) = \cos(K \sin x)$
where $K \in R^+$ then,
If $K \in [0, \pi)$ $\Rightarrow$ Range is $y \in [\cos K, 1]$
If $K \in [\pi, \infty)$ $\Rightarrow$ Range $\rightarrow y \in [-1, 1]$

Illustration 5

Find range of the function $y = \cos(2 \sin x)$

Short-cut solution :

$\boxed{\text{Using T-6}}$ $\because$ $K = 2$ (case IInd) and $K < \pi$

$\Rightarrow$ Range $\rightarrow y \in [\cos 2, 1]$

Illustration 6

Find range of the function $y = \cos(3 \sin x)$

Short-cut solution :

$\boxed{\text{Using T-6}}$ $\because$ $K = 3$ $\Rightarrow$ $K < \pi$ (case IInd)

$\Rightarrow$ Range $\rightarrow y \in [\cos 3, 1]$

Illustration 7

Find the range of the function $f(x) = \cos(4 \sin x)$

Short-cut solution :

$\boxed{\text{Using T-6}}$ $\because$ $K = 4$ $\Rightarrow$ $K > \pi$ (case Ist)

Hence, Range $\rightarrow y \in [\cos -1, 1]$

TIPS AND TRICKS: (T-7)

Identification of function using graph:

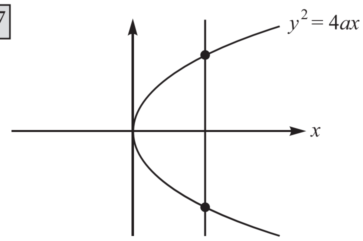
If it is possible to draw lines parallel to y-axis which cuts the curve more than one point then the given relation is not a function and when the line cuts the curve at only one point than it is a function.

Illustration 8

Check whether it is a function or not $y^2 = 4\,ax$

Short-cut solution :

Using T-7



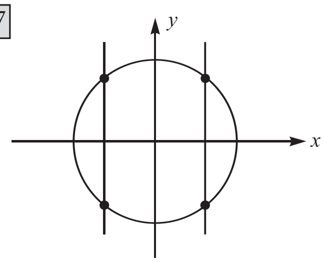
Vertical line cuts the graph more than one time then it is not a function.

Illustration 9

Check whether $x^2 + y^2 = a^2$ is a function or not

Short-cut solution :

Using T-7



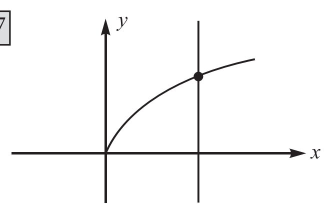
$\Rightarrow$ It is not a function.

Illustration 10

Check whether $y = \sqrt{x}$ is a function or not.

Short-cut solution :

Using T-7



$\Rightarrow$ It is a function since it cuts the curve once.

TIPS AND TRICKS: (T-8(i))

Graphical approach to check one-one or many-one function

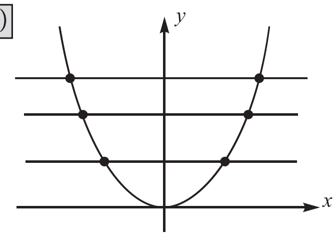
Construct the graph and draw lines parallel to x-axis, if it cuts the graph one time then it is a function and if it cuts more than one time then it is a many-one function.

Illustration 11

Check whether $y = x^2$ is a one-one or many-one function.

Short-cut solution :

$\boxed{\text{Using T-8(i)}}$



It is a many-one function.

TIPS AND TRICKS: (T-8(ii))

Calculus approach to check one-one or many-one function

Differentiate the function. $y = f(x)$ $\{f(x)$ must be differentiable$\}$

★ If $\dfrac{dy}{dx} > 0 \Rightarrow$ Monotonically Increases $\Rightarrow$ one-one function

★ If $\dfrac{dy}{dx} < 0 \Rightarrow$ Monotonically Decreases $\Rightarrow$ one-one function

★ If $\dfrac{dy}{dx} > 0 \Rightarrow$ for some 'x' and $\dfrac{dy}{dx} > 0$ for some x then function is many-one function.

Illustration 12

Check whether one-one or many-one function.

(i) $f(x) = x^3$ (ii) $f(x) = x^2$

Short-cut solution :

$\boxed{\text{Using T-8(ii)}}$

(i) $f(x) = x^3 \Rightarrow f'(x) = 3x^2 > 0 \Rightarrow$ one-one function

(ii) $f(x) = x^2 \Rightarrow f'(x) = 2x \left\langle \begin{array}{l} x > 0 \Rightarrow \uparrow \text{ses} \\ x < 0 \Rightarrow \downarrow \text{ses} \end{array} \right\}$ Many- one function

★**Note:** (1) All trigonometric functions are many-one function.

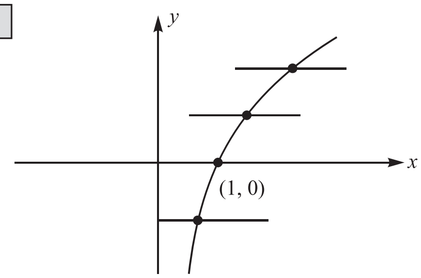
(2) All inverse trigonometric function are one-one function.

Illustration 13

Check for one-one or many-one function for $f(x) = \log_e x$

Short-cut solution :

Using T-8(ii)



It is one-one function.

Illustration 14

Check whether the following functions $f(x)$ are either one-one or many-one function.

(i) $f(x) = e^x$

(ii) $f(x) = \sin x$

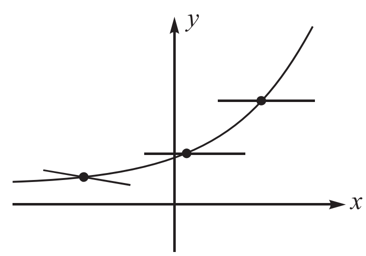
(iii) $f(x) = |x|$

(iv) $f(x) = \tan x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Short-cut solution :

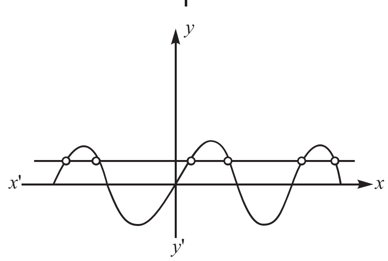
Using T-8(ii)

(i) $f(x) = e^x$



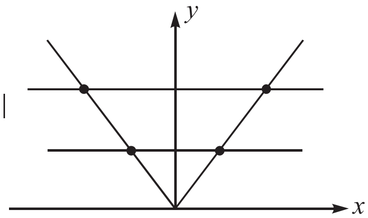
One-one function.

(ii) $f(x) = \sin x$



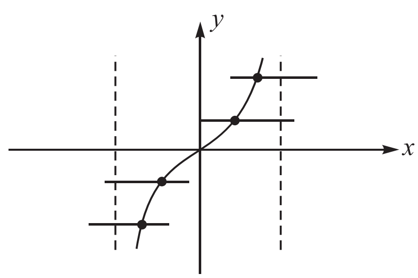
$\Rightarrow$ Many-one function.

(iii) $f(x) = |x|$



$\Rightarrow$ Many-one function.

(iv) $f(x) = \tan x$ in $x \in \left(-\dfrac{\pi}{2}, -\dfrac{\pi}{2}\right)$



$\Rightarrow$ One-one function.

Illustration 15

Check whether one-one or many-one function for

$f(x) = f : R \rightarrow R$

$f(x) = x^3 + x^2 + 7x + \sin x$

Short-cut solution :

$\boxed{\text{Using T-8(ii)}}$

Differentiate $\quad f'(x) = 3x^2 + 2x + 7 + \cos x$

$$= (x^2 + 2x + 1) + (2x^2 + 6) + \cos x$$

$$f'(x) = \underbrace{(x+1)^2}_{\geq 0} + \underbrace{(2x^2 + 6)}_{\geq 6} + \underbrace{\cos x}_{[-1,1]}$$

$\Rightarrow \quad f'(x) > 0 \Rightarrow$ one-one function

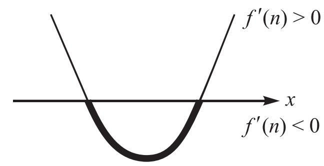
Illustration 16

Check whether one-one or many-one function for

$f(x) = x^3 + 6x^2 + 11x$

Short-cut solution :

Using T-8(ii)



Differentiate $f'(x) = 3x^2 + 12x + 11$

This is a parabola open upwards

Now, we find discriminant.

$D = 144 - 132$

$\Rightarrow \quad D > 0$ (two distinct roots)

So this implies $f'(x) > 0$ and $f'(x) < 0$ both.

$\Rightarrow$ Many-one function.

TIPS AND TRICKS: (T-9)

Short trick to check whether onto or into function for polynomial functions. For $x \in R$

Put, $x \to \infty$ If $\Rightarrow f(x) \to \infty$

Put, $x \to -\infty$ If $\Rightarrow f(x) \to -\infty$

Hence, onto function.

Illustration 17

Check whether onto or into function.

(i) $f(x) = x^3$

(ii) $f(x) = x^2$

Short-cut solution :

Using T-9

(i) Since $f(x)$ is odd function.

$$\Rightarrow \left.\begin{array}{l} x \to +\infty, \quad f(x) \to +\infty \\ x \to -\infty, \quad f(x) \to -\infty \end{array}\right\} \Rightarrow \text{One- one function and onto function.}$$

(ii) $f(x) = a_0 x^{2n} + a_1 x^{2n-2} + \ldots\ldots\ldots$

$$\text{Put, } \left.\begin{array}{l} x \to +\infty, \quad f(x) \to \infty \\ x \to -\infty, \quad f(x) \to \infty \end{array}\right\} \Rightarrow \text{Many- one function and into function.}$$

Illustration 18

Check whether the given function is bijective or not

$f(x) = x^3 + 5x + 1$ **[AIEEE 2009]**

Short-cut solution :

Using T-8(ii)

On differentiating $\to f(x) = 3x^2 + 5 > 0$s

$\Rightarrow$ one-one function.

$\boxed{\text{Using T-9}}$

$$\left.\begin{array}{l} x \to +\infty, \quad f(x) \to \infty \\ x \to -\infty, \quad f(x) \to -\infty \end{array}\right\} \Rightarrow \text{Onto function}$$

Hence, above function is bijective.

TIPS AND TRICKS: (T-10)

If set A contains 'm' elements and another set B contains 'n' elements, then the

(i) Total number of functions = $(n)^m$

For $f : A \to B$

(ii) Number of one-one function is $\boxed{^nP_m}$ for $n \geq m$ and 0 (zero) for $n < m$

(iii) Number of many-one function = Total number of functions – One-one function

(iv) Number of onto functions are

(a) If $n < m \Rightarrow n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + ...$

(b) If $n = m \Rightarrow n!$

(c) If $n > m \Rightarrow 0$

(v) Number of into function are

(a) If $n \leq m \Rightarrow$ Total number of functions – Onto functions

(b) If $n > m \Rightarrow (n)^m$ [Total – Onto]

(vi) Number of constant functions = n.

(vii) If A and B are two sets having n-elements and '2' elements respectively.

Then number of onto functions from A to B

is $2^n - 2$ if $n \geq 2$ and '0' if $n > 2$

Illustration 19

If $A = \{1, 5, 9, 7, 14, 22\}$ and $B = \{2, 3, 5, 6\}$, then number of:

(a) Total functions

(b) One-one functions

(c) Many-one functions

(d) Onto function

(e) Into functions

(f) Constant functions

Short-cut solution :

Since, $m = 6$ and $n = 4$

(a) $\boxed{\text{Using T-10(i)}}$ $\Rightarrow (n)^m = (4)^6 = 4096$

(b) $\boxed{\text{Using T-10(ii)}}$ $\Rightarrow$ Since $n < m \Rightarrow$ One-one function = 0

(c) $\boxed{\text{Using T-10(iii)}}$ $\Rightarrow$ Many-one function = Total – (One-one functions)

$= 4096 - 0 = 4096$

(d) $\boxed{\text{Using T-10(iv)}}$ Since $n < m$

$= (n^m) - {}^nC_1\,(n-1)^m + {}^nC_2\,(n-2)^m + \ldots\ldots$

$= 4^6 - {}^4C_1 \cdot 3^6 + {}^4C_2 \cdot (2)^6 - {}^4C_3 \cdot (1)^6 + {}^4C_4 \cdot (0)^6$

$= 4096 - 4 \times 729 + 6 \times 64 - 4 + 0$

$= 1560$

(e) $\boxed{\text{Using T-10(v)}}$ Total – Onto

$4096 - 1560 = 2536$

(f) $\boxed{\text{Using T-10(vi)}}$ Constant function $= n = 4$

TIPS AND TRICKS: (T-11)

If A and B are finite sets and $f : A \to B$ is a bijection, then A and B have same number of the elements. If A has n elements, then number of the bijections from A to B is $\boxed{n!}$

Illustration 20

If $A = \{2, 3, 4, 5, 6\}$, then the total number of bijection function in $f : A \to A$ is

(a) 110 (b) 115 (c) 120 (d) 125

Short-cut solution :

$\boxed{\text{Using T-11}}$ Since $n = 5$

$\Rightarrow 5! = 120$ **Ans. (c)**

TIPS AND TRICKS: (T-12)

In order to find fundamental period of

(i) $\sin(2n\pi\{x\}) \longrightarrow \boxed{\dfrac{1}{n}}$; where $\{x\}$ is fractional part function and $n \in I^+$

(ii) $\sin(2n+1)\pi\{x\} \longrightarrow \boxed{1}$

(iii) If $f(x)$ is periodic function with fundamental period 'T' then $\dfrac{}{f(x)}$ and $\sqrt{f(x)}$ will also be periodic with fundamental period 'T'.

Illustration 21

Find the fundamental period of

(i) $\sin(2\pi(\{x\}))$ (ii) $\sin(4\pi\{x\})$

Short-cut solution :

$\boxed{\text{Using T-12(i)}}$ $\dfrac{1}{n} = \dfrac{1}{1} \Rightarrow \boxed{T = 1}$

(ii) sin (4π {x})

Using T-12(ii) $\frac{1}{n} = \frac{1}{2} \Rightarrow \boxed{T = \frac{1}{2}}$

Illustration 22

Find the fundamental period of $f(x) = \sec x$

Short-cut solution :

Using T-12(i) $f(x) = \frac{1}{\cos x}$

Since period of $\cos x$ is 2π

Hence period of $f(x) = \sec x$ is also 2π.

Illustration 23

Find the fundamental period of $f(x) = \frac{1}{\sqrt{\tan x}}$

Short-cut solution :

Using T-12(iii) $f(x) = \sqrt{\cot x}$

Since fundamental period of $\cot x$ = is 'π'

Then, fundamental period of $\sqrt{\cot x} = \frac{1}{\sqrt{\tan x}}$ is also 'π'

TIPS AND TRICKS: (T-13)

If $f(x)$ is periodic with fundamental period 'T' than $f(ax + b)$ is also periodic with fundamental period $\frac{T}{|a|}$

Illustration 24

Find the fundamental period of $f(x) = \sin(2x + 3)$

Short-cut solution :

Using T-13 Since fundamental period of $\sin x$ is 2π

$\Rightarrow T = \frac{2\pi}{|2|} \Rightarrow T = \pi$

Illustration 25

Find the period of $f(x) = \{-3x + 5\}$ where {*} is fractional part function.

Short-cut solution :

Using T-13 Since period of $y = \{x\}$ is 1

Hence, $T = \frac{1}{|-3|} \Rightarrow T = \frac{1}{3}$

TIPS AND TRICKS: (T-14)

Let $f(x)$ and $g(x)$ be the two functions which are periodic then period of $h(x) = f(x) + g(x)$ is $\boxed{\text{LCM of } T_1 \text{ and } T_2}$ as one its periods.

where, T_1, T_2 are period of $f(x)$ and $g(x)$ respectively.

★**Note:** LCM of irrational number $= \dfrac{\text{LCM of numerator}}{\text{HCF of denominator}}$

Illustration 26

Find period of $f(x) = \cos(\sin x) + \cos(\cos x)$

Short-cut solution :

$\boxed{\text{Using T-14}}$ Since, period of $\cos(\sin x)$ is π and priod of $\cos(\cos x)$ is π

Hence the priod of $f(x) = \text{LCM}(\pi, \pi) = \pi$.

Illustration 27

Find period of $f(x) = -\sin\dfrac{2x}{7} + \cos\dfrac{3x}{5}$.

Short-cut solution :

$\boxed{\text{Using T-13}}$ Since period of $-\dfrac{\sin 2x}{7}$ is 7π

and period of $\dfrac{\cos 2x}{5}$ is $\dfrac{10\pi}{3}$

$\boxed{\text{Using T-14}}$

Hence, the period of $f(x) = \text{LCM}\left(7\pi, \dfrac{10\pi}{3}\right)$

$$\Rightarrow \frac{\text{LCM of } N^r}{\text{HCF of } D^r} = \frac{\text{LCM}(7\pi, 10\pi)}{\text{HCF}(1,3)} = \frac{70\pi}{1}$$

TIPS AND TRICKS: (T-15)

Let $y = f(x)$ is a function and it satisfies the relation $f(n+a) + f(n+b) = $ constant then period of this junction is $2|b-a|$.

Illustration 28

If $f(x) + f(x+5) = 12$, the period of $f(x)$ is:

Short-cut solution :

Using T-15 $2|5-0| = 10$.

Illustration 29

If $f(x+2) + f(x+9) = 30$, then period of $f(x)$ is:

Short-cut solution :

Using T-15 $2|9-2| = 14$.

TIPS AND TRICKS: (T-16)

$$[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx]$$

where, $n \in N$ and [*] is greatest integer function.

Illustration 30

Find the value of

$$\left[\frac{1}{5}\right] + \left[\frac{1}{5} + \frac{1}{100}\right] + \left[\frac{1}{5} + \frac{2}{100}\right] + \ldots\ldots + \left[\frac{1}{5} + \frac{99}{100}\right]$$

Short-cut solution :

Using T-16 $[nx] = \left[100 \times \frac{1}{5}\right] = 20$

SHORTCUTS: (SC-1)

To find number of solution of $f(x) = g(x)$ where $f(x)$ and $g(x)$ are functions. Then draw graph for both LHS and RHS on same Cartesian plane and find number of point of intersection.

Number of point of intersection = Number of solutions/ Roots

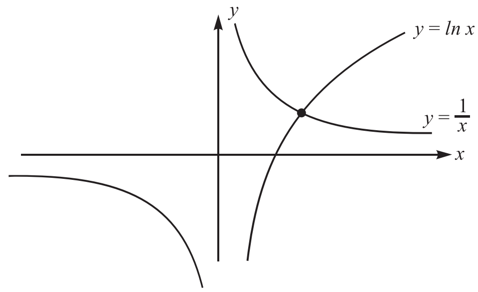
Illustration 31

Find number of solution of $x \cdot ln\, x = 1$

Short-cut solution :

Using SC-1 $\because \quad ln\, x = \frac{1}{x}$; $x > 0$ and $x \neq 0$

Now, we draw graph for $y = ln\, x$ and $y = \frac{1}{x}$; $x > 0$



Number of intersection = 1 = Number of solutions

Illustration 32

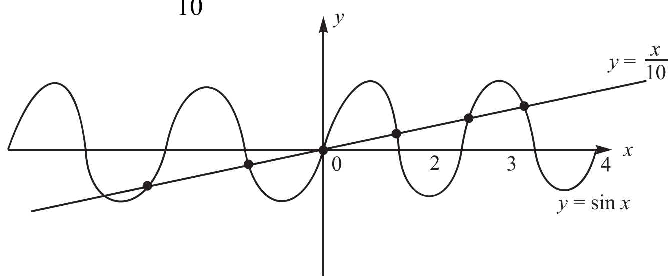
Find number of solutions of $\sin x = \dfrac{x}{10}$.

Short-cut solution :

$\boxed{\text{Using SC-1}}$ Now, we will draw graphs for $y = \sin x$ and $y = \dfrac{x}{10}$

$\because \quad \sin x \in [-1, 1]$

$\Rightarrow \quad -1 \le \dfrac{x}{10} \le 1 \quad \Rightarrow \quad -10 \le x \le 10$



We have to sketch the curve when $x \in [-10, 10]$

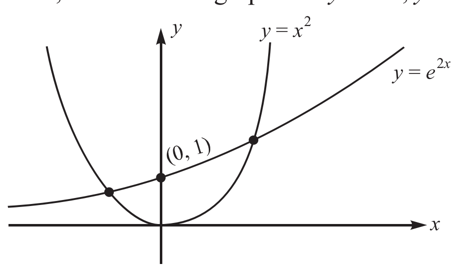
Hence, number of intersection = 7 = Number of solutions.

Illustration 33

Find the number of solutions of $e^{2x} = x^2$.

Short-cut solution :

$\boxed{\text{Using SC-1}}$ Now, we will draw graphs for $y = e^{2x}$, $y = x^2$



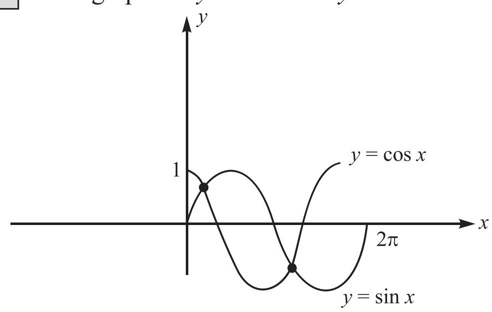
Number of intersection = Number of solutions = 2.

Illustration 34

Find the number of solutions of $\sin x = \cos x$ in $x \in [0, 2\pi]$.

Short-cut solution :

Using SC-1 Draw graphs of $y = \sin x$ and $y = \cos x$



Number of intersection in $x \in [0, 2\pi] = 2$ = Number of solutions

SHORTCUTS: (SC-2)

In order to find domain and range of function, the shortest way is to draw graph. Graphs are also useful for one-one, many-one, onto and into.

Domain = Existence of graph along x - axis
Range = Existence of graph along y - axis

Illustration 35

Find the domain and range also check function is one-one or not in its domain. $f(x) = \dfrac{x^2 - 5x + 4}{x^2 + 2x - 3}$

Short-cut solution :

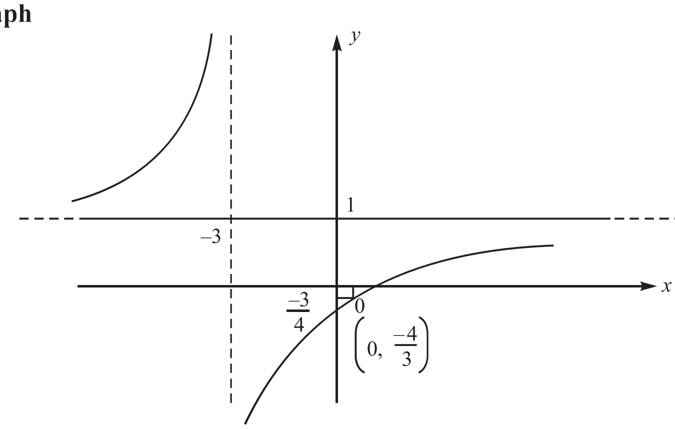
Using SC-2 $f(x) = \dfrac{x^2 - 5x + 4}{x^2 + 2x - 3} = \dfrac{(x-4)(x-1)}{(x+3)(x-1)}; \quad x \neq 1, -3$

First of all we will check monotonicity.
Differentiating $f(x)$ w.r.t. x, we get

$$\Rightarrow \quad f'(x) = \frac{d}{dx}\left(\frac{x-4}{x+3}\right) = \frac{(x+3)-(x-4)}{(x+3)^2} = \frac{7}{(x+3)^2}$$

$$\Rightarrow \quad f'(x) > 0 \quad \Rightarrow \quad \text{Increasing function}$$

Graph



$\Rightarrow$ Domain : $x \in R - \{-3, 1\}$

Range : $x \in R - \left\{1, \frac{-3}{4}\right\}$

As shown function is **one-one**.

Illustration 36

$f : [2, \infty] \to Y$

$f(x) = x^2 - 4x + 5$ is both one-one and onto if

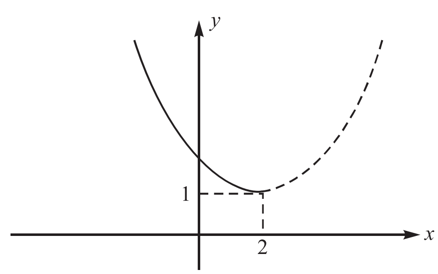
(a) $Y = R$ (b) $Y = [1, \infty)$

(c) $Y = [4, \infty)$ (d) $Y = [5, \infty)$

Short-cut solution :

$\boxed{\text{Using SC-2}}$ Rewrite, $f(x) = (x - 2)^2 + 1 \Rightarrow$ Parabola with vertex (2, 1)

Graph



$y_{min} = 1$; $y_{max} = \infty$

Hence, $Y = [1, \infty)$.

Illustration 37

$f : R \to R$

$$f(x) = \begin{cases} x^2 + 2mx - 1 \,;\, x \le 0 \\ mx - 1 \,;\, x > 0 \end{cases}$$ then find the value of m

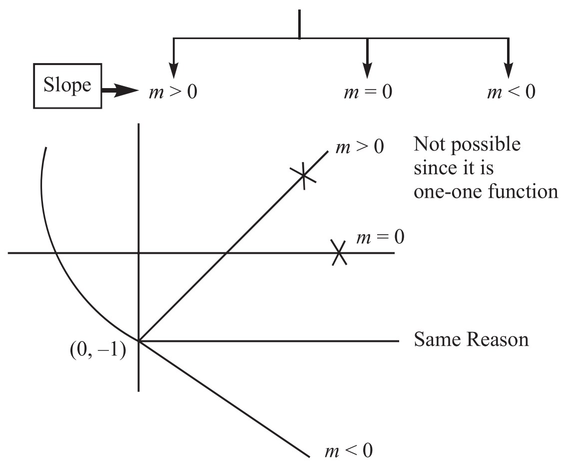
Short-cut solution :

$\boxed{\text{Using SC-2}}$

Graph

For $x < 0$, it is parabola open upwards

For $x > 0$, there are three possibilities (line)



Hence, $m < 0 \Rightarrow m \in (-\infty, 0)$.

SHORTCUTS: (SC-3)

Inverse of many-one function does not exist, only one-one onto function are invertible.

$$f'(x) \ge 0 \quad \text{or} \quad f(x) \le 0$$

Illustration 38

If $f : R \to R,\ f(x) = x^3 + (a + 2)\, x^2 + 3ax + 5$ is invertible mapping. Find 'a'.

Short-cut solution :

$\boxed{\text{Using SC-3}}$ Invertible $\Rightarrow$ One-one + Onto

Hence $f'(x) \ge 0$ or $f'(x) \le 0$

$\Rightarrow$ $f'(x) = 3x^2 + 2x\,(a + 2) + 3a \ge 0$ or $\le 0\ \forall\ n \in R$

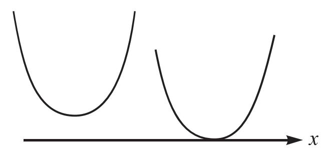
$\Rightarrow \quad \boxed{D \le 0}$

Hence, $4(a+2) - 4.9a \le 0$

$$a^2 - 5a + 4 \le 0$$

$$(a-1)(a-4) \le 0$$

$\Rightarrow \quad a \in [1, 4]$



SHORTCUTS: (SC-4)

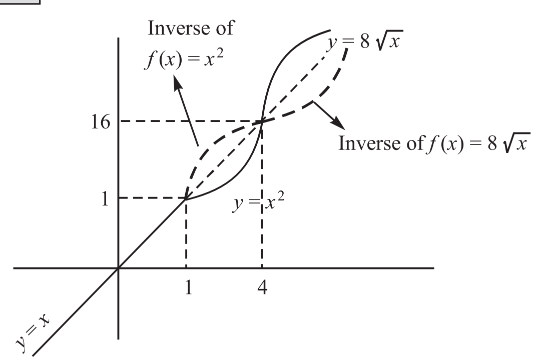
Inverse of a function is a mirror image about $y = x$ line. So copy the graph along other side of $y = x$ line.

Illustration 39

If $f(x) = \begin{cases} x & ; \quad x < 1 \\ x^2 & ; \quad 1 \le x \le 4 \\ 8\sqrt{x} & ; \quad x > 4 \end{cases}$. Then find $f^{-1}(x)$

Short-cut solution :

Using SC-4 **Draw Graph**



Hence, $f^{-1}(x) = \begin{cases} x & ; \quad x < 1 \\ \sqrt{x} & ; \quad 1 \le x \le 16 \\ \dfrac{x^2}{64} & ; \quad x > 16 \end{cases}$

SHORTCUTS: (SC-5)

Even functions are symmetric about y-axis odd functions are symmetric about origin.

Illustration 40

If $f(x) = \begin{cases} x^2 & ; \ x \in (0, 1] \\ 2 - x ; & x > 1 \end{cases}$. Define $f(x)$ for $x < 0$ if $f(x)$ is

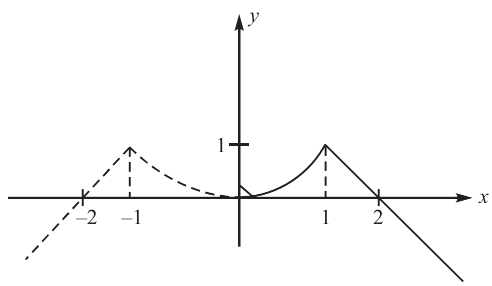
(i) Even function (ii) Odd function

Short-cut solution :

Using SC-5

(i) **Even Function**

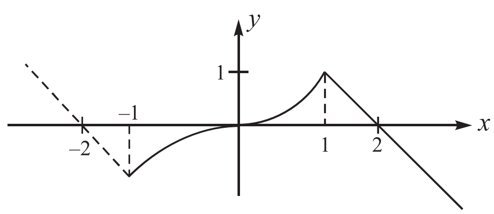
Draw graph of $f(x)$ using short cut $\Rightarrow$ Symmetry about y-axis



Hence, $f(x) = \begin{cases} x + 2 \ ; & x \in (-\infty, -1) \\ x^2 \ ; & x \in (-1, 0) \end{cases}$

(ii) **Odd Function**

Draw graph of $f(x)$ using shortcut $\Rightarrow$ Symmetry about origin



Hence, $f(x) = \begin{cases} -x - 2 \ ; & x \in (-\infty, -1) \\ -x^2 \ ; & x \in (-1, 0) \end{cases}$

TECHNIQUE

If f^{-1} be the inverse of bijective function $f(x)$ then $f(f^{-1}(x)) = x$.
Apply the formula of f on $f^{-1}(x)$ and use of the identity $f(f^{-1}(x)) = x$ to solve for $f^{-1}(x)$

Illustration 41

Find the inverse of the function $f(x) = \log_a\left(x + \sqrt{x^2 + 1}\right); a > 1$

Short-cut solution :

Using Tech.

$$f\left(f^{-1}(x)\right) = x$$

$$\Rightarrow \log_a\left(f^{-1}(x) + \sqrt{\left(f^{-1}(x)\right)^2 + 1}\right) = x$$

$$\Rightarrow f^{-1}(x) + \sqrt{\left(f^{-1}(x)\right)^2 + 1} = a^x \quad ...(i)$$

and $-f^{-1}(x) + \sqrt{\left(f^{-1}(x)\right)^2 + 1} = a^{-x}$ ..(ii)

From (i) and (ii), $f^{-1}(x) = \left(\dfrac{a^x - a^{-x}}{2}\right)$

Concept Booster Exercise

1. Find a polynomial of degree '5' which satisfies the relation $f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right)$ which is always decreasing function.

2. Find a polynomial which satisfies $f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right)$ $\forall\, x \in R-\{0\}$ and the condition $f(3) = -26$, then determine $f'(1)$.

3. Find: $I = \int_0^{3\pi} \{2\cos x\}\, dx$; where {} is fractional part function.

4. The range of the function $f(x) = {}^{7-x}P_{x-3}$ is **[AIEEE 2004]**

(a) {1, 2, 3} (b) {1, 2, 3, 4, 5, 6} (c) {1, 2, 3, 4} (d) {1,2,3,4,5}

5. If $f: R \to S$, defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$, is onto, then the interval of 'S' is: **[AIEEE 2004]**

(a) [0, 3] (b) [–1, 1] (c) [0, 1] (d) [–1, 3]

6. The range of the function $f(x) = \frac{2+x}{2-x}$, $x \neq 2$ is **[AIEEE 2002]**

(a) R (b) $R - \{-1\}$ (c) $R - \{1\}$ (d) $R - \{2\}$

7. The range of $f(x) = \frac{3x-1}{2x+1}$; $x \neq \frac{-1}{2}$

(a) $R - \left\{\frac{2}{3}\right\}$ (b) $R - \left\{\frac{3}{2}\right\}$ (c) R (d) $R - \left\{\frac{2}{5}\right\}$

8. The range of the function $f(x) = \cos\left(\frac{1}{2}\sin x\right)$

(a) [cos 2, 1] (b) $\left[\cos\frac{1}{2}, 1\right]$

(c) [–1, 1] (d) None of these

9. The range of the function $f(x) = \cos(5\sin x)$

(a) [cos 5, 1] (b) [–1, 1] (c) $\left[\cos\frac{1}{5}, 1\right]$ (d) [–1, 1)

10. Which of the following is/are the functions **[AIEEE 2002]**

(a) $y^2 = 4x$ (b) $x^2 = 8y$ (c) $x^2 + y^2 = 4$ (d) $\left[\cos\frac{1}{5}, 1\right]$

11. The function $f: R \to R$ defined by $f(x) = \sin x$ is:

(a) Into (b) Onto (c) One-one (d) Many-one

12. The period of the function $f(x)$ = if $f(x+10)+f(x+20)=50$

(a) 30 (b) 40 (c) 60 (d) 20

13. The period of $f(x) = \cos\frac{\pi}{4}x + \sin\frac{\pi x}{3}$.

(a) 24 (b) 12 (c) 36 (d) 6

14. The function $f: R \to R$ defined by $f(x) = x^2 - 3x + 2$

(a) Onto (b) Into (c) Many-one (d) One-one

15. Given $X = \{1, 2, 3, 4\}$, find all one-one, onto mappings, $f: X \to X$ such that, ...

16. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. The number of onto functions from E to F is **[AIEEE 2002]**

(a) 14 (b) 16 (c) 12 (d) 8

17. Suppose $f(x) = (x+1)^2$ for $x \geq -1$, If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals **[AIEEE 2002]**

(a) $-\sqrt{x}-1, x \geq 0$ (b) $\frac{1}{(x+1)^2}, x \geq -1$

(c) $\sqrt{x+1}, n \geq -1$ (d) $\sqrt{x}-1, x \geq 0$

18. Let $f: R \to R$ be defined by $f(x) = 2x + \sin x$; $x \in R$. Then f is **[AIEEE 2002]**

(a) one to one and onto (b) one to one but not onto

(c) onto but not one-one (d) neither one-to one nor onto

19. The function $f: [0, 3] \to [1, 29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$ is **[AIEEE 2012]**

(a) one-one and onto (b) onto but not one-one

(c) one-one but not onto (d) neither one-one nor onto

20. The inverse function of $f(x) = \frac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}$, $x \in (-1, 1)$ is **[JEE M 2020]**

(a) $\frac{1}{4}\log_e\left(\frac{1+x}{1-x}\right)$ (b) $\frac{1}{4}\log_e\left(\frac{1-x}{1+x}\right)$

(c) $\frac{1}{4}\left(\log_8^e\right)\log_e\left(\frac{1+x}{1-x}\right)$ (d) $\frac{1}{4}\left(\log_8^e\right)\log_e\left(\frac{1-x}{1+x}\right)$

21. Let $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \rightarrow R$ given by $f(x) = [\log(\sec x + \tan x)]^3$, then **[JEE M 2014]**

(a) $f(x)$ is odd function (b) $f(x)$ is an one-one function

(c) $f(x)$ is onto function (d) $f(x)$ is even function

22. Let $f(x) = ax^7 + bx^3 + cx - 5$; $a, b, c \in$ constant then find $f(+7)$, if $f(-7) = 7$.

(a) – 16 (b) – 15

(c) – 17 (d) – 20

23. Find the period of the function $f(x) = 2 + 3\cos(3x - 2)$

(a) 2π (b) π (c) — (d) $\frac{\pi}{3}$

24. Find the domain of the function $f(x) = \dfrac{\sqrt{\cos - \frac{1}{\sqrt{2}}}}{\sqrt{\frac{3}{2}x - x^2 - \frac{1}{2}}}$

25. Let $f(x) = \dfrac{x}{1+x}$ defined from $(0, \infty) \rightarrow [0, \infty)$ then $f(x)$ is **[AIEEE 2003]**

(a) one-one but not onto (b) one-one and onto

(c) many-one but not onto (d) many-one and onto

Solutions

1. **$(1-x^5)$** $\boxed{\text{Using T-1}}$ $f(x) = 1 \pm x^5$

Since function is always decreasing is

So, $f'(x) < 0$

$\Rightarrow f(x) = 1 + x^5$ or $f(x) = 1 - x^5$

$f'(x) = 5x^4 > 0$ or $f'(x) = -5x^4 < 0$

$\Rightarrow$ increasing function $\Rightarrow$ decreasing function

Hence, $f(x) = 1 - x^5$ is our required answer.

2. **(–3)** $\boxed{\text{Using T-1}}$ $f(x) = 1 \pm x^n$

$\because$ $f(3) = -26$

$f(3) = 1 + 3^n$		$f(3) = 1 - 3^n$
$-26 = 1 + 3^n$		$-26 = 1 - 3^n$
$-27 = 3^n$	$\Rightarrow$	$3^n = 27$
Not possible	$\Rightarrow$	$\boxed{n = 3}$

Hence, the required function is

$f(x) = 1 - x^3 \Rightarrow f'(x) = -3x^2$

$\Rightarrow f'(1) = -3$

3. $\left(\dfrac{3\pi}{2}\right)$ $\boxed{\text{Using T-3}}$

$$I = \int_0^{3\pi} \{-2\cos x\}\, dx \qquad \text{... (1)}$$

Apply king property

$$\Rightarrow I = \int_0^{3\pi} \{2\cos x\}\, dx \qquad \text{... (2)}$$

Add eqns. (1) and (2)

$$\Rightarrow 2I = \int_0^{3\pi} (\{2\cos x\} + \{-2\cos x\})\, dx$$

$$\Rightarrow 2I = \int_0^{3\pi} (1)\, dx \Rightarrow I = \frac{3\pi}{2}$$

4. **(a)** We know that $x - 3 \ge 0 \Rightarrow x \ge 3$

And $7 - x \ge x - 3 \Rightarrow 2x \le 10 \Rightarrow x \le 5$

Hence, $x = 3, 4, 5$

Now At $x = 3 \Rightarrow y = {}^4P_0 = 1$

At $x = 4 \Rightarrow y = {}^3P_1 = 3$

At $x = 5 \Rightarrow y = {}^2P_2 = 2$

Hence, range = $\{1, 2, 3\}$.

5. **(d)** As we know that $-\sqrt{a^2+b^2} \le a \sin x + b \cos x \le \sqrt{a^2+b^2}$

Hence, $\qquad -2 \le \sin x - \sqrt{3}\cos x \le 2$

$\qquad -1 \le \sin x - \sqrt{3}\cos x + 1 \le 3$

Onto $\Rightarrow$ Range = Codomain $\Rightarrow$ $S \in [-1, 3]$.

6. **(b)** $\boxed{\text{Using T-5}}$ Range $\rightarrow y \in R - \left\{\frac{1}{-1}\right\}$

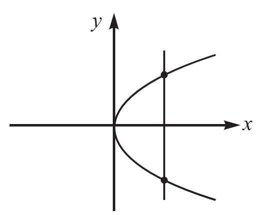
$\Rightarrow \quad y \in R - \{-1\}$

7. **(b)** $\boxed{\text{Using T-5}}$ Range $\rightarrow y \in R - \left\{\frac{3}{2}\right\}$

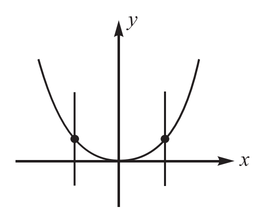
8. **(b)** $\boxed{\text{Using T-6}}$ $\because \frac{1}{2} < \pi \Rightarrow$ Range is $y \in \left[\cos\frac{1}{2}, 1\right]$

9. **(b)** $\boxed{\text{Using T-6}}$ $\because 5 > \pi \Rightarrow$ Range is $y \in [-1, 1]$

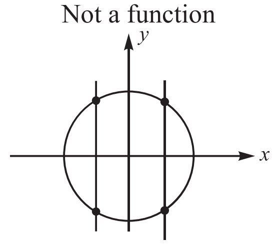
10. **(b, d)** $\boxed{\text{Using T-7}}$

(a) 

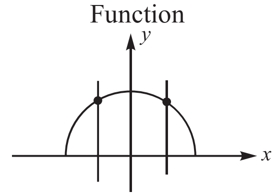
Not a function

(b) 

Function

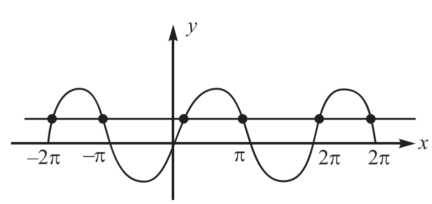
(c) 

Not a function

(d) 

Function

11. **(a, d)** $\boxed{\text{Using T-8}}$



$\Rightarrow$ Many-one

$\left.\begin{array}{l}\text{Range of } \sin x = y \text{ is } y \in [-1, 1] \\ \text{But given co-domain is } y \in R\end{array}\right\} \Rightarrow \text{Range} \ne \text{Codomain}$

Hence, into function.

12. **(d)** Using T-15 $a = 10, \quad b = 20$

$\Rightarrow$ Period $= 2\,|20 - 10| = 20$.

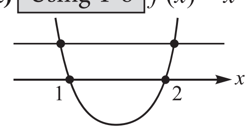
13. **(a)** $f(x) = \underbrace{\sin\frac{\pi x}{4}}_{T_1} + \underbrace{\sin\frac{\pi x}{3}}_{T_2}$, Since period of $\sin x = 2\pi$

Using T-13 $T_1 = \dfrac{2\pi}{\frac{\pi}{4}}$ and $T_2 = \dfrac{2\pi}{\frac{\pi}{3}}$

Using T-14 Period = LCM (T_1, T_2)

= LCM (8, 6) = 24.

14. **(b, c)** Using T-8 $f(x) = x^2 - 3x + 2 \;\Rightarrow$ Parabola open upwards



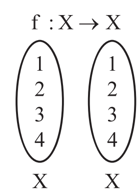
$\Rightarrow$ Many-one function

Using T-9 $\left.\begin{array}{l} x \to \infty \;\;\Rightarrow\;\; f(x) \to \infty \\ x \to -\infty \;\Rightarrow\; f(x) \to \infty \end{array}\right\} \Rightarrow$ Into function

15. **(24)** Using T-10(ii) $m = 4$ and $n = 4$

$^nP_m = {}^4P_4 = \dfrac{4!}{0!}$

$n = m$



$\Rightarrow$ No. of one-one functions = 24

Using T-10(iv) Since, $n = m = 4$

$\Rightarrow$ No. of onto functions $= n! = 4! = 24$.

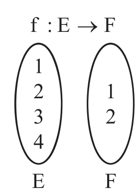
16. **(a)** Using T-10(iv) $m = 4$ and $n = 2$

Since, $n < m$

$\Rightarrow n^m - {}^nC_1\,(n-1)^m + {}^nC_2\,(n-2)^m - {}^nC_3\,(n-3)^m + \ldots$

$= \; 2^4 - {}^2C_1\,(1)^4 + {}^2C_2 \times 0$

$= 16 - 2 = 14$.



17. **(d)** Using SC-3

Since, $g(x)$ is reflection about $y = x$ line of $f(x)$

$\Rightarrow g(x)$ is inverse of $f(x)$

Hence, $y = (x+1)^2 \;\Rightarrow\; x + 1 = \sqrt{y} \;\Rightarrow\; x = -1 + \sqrt{y}$

$\Rightarrow f^{-1}(x) = -1 + \sqrt{x}\;;\; x \geq 0$.

18. **(a)** $\boxed{\text{Using T-8(ii)}}$ Differentiate $f(x)$

$\Rightarrow \quad f'(x) = 2 + \underbrace{\cos x}_{[-1,1]} \Rightarrow f'(x) > 0 \Rightarrow$ one-one function

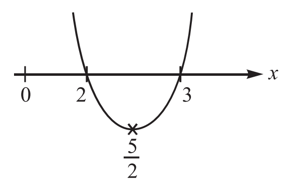
$\boxed{\text{Using T-9}}$ $f(x) = 2x + \underbrace{\sin x}_{[-1,1]}$

$$\left.\begin{array}{ll} x \to \infty & \Rightarrow f(x) \to \infty \\ x \to -\infty & \Rightarrow f(x) \to -\infty \end{array}\right\} \Rightarrow \text{Onto function.}$$

19. **(d)** $\boxed{\text{Using T-8(ii)}}$ Differentiate $f(x)$

$f'(x) = 6x^2 - 30x + 36$

$= 6(x^2 - 5x + 6)$



$$\left.\begin{array}{ll} \text{For, } x \in [0, 2] & \Rightarrow f'(x) > 0 \\ \text{For, } x \in [2, 3] & \Rightarrow f'(x) < 0 \end{array}\right\}$$

Hence, many-one function.

20. **(c)** $\frac{y}{1} = f(x) = \frac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}; x \in (-1, 1)$

Apply componendo and dividendo

$\Rightarrow \frac{y+1}{y-1} = \frac{2.8^{2x}}{-2.8^{-2x}} = -8^{4x}$

Take $\log_8$ to both sides

$\Rightarrow 4x = \log_8\left(\frac{y+1}{1-y}\right) \Rightarrow f^{-1}(x) = \frac{1}{4}\log_8\left(\frac{y+1}{1-y}\right)$ {Change base}

$\Rightarrow f^{-1}(x) = \frac{1}{4}\frac{\log_e\left(\frac{x+1}{1-x}\right)}{\log_e 8}$

21. **(a, b, c)** For even/odd function

$\Rightarrow f(-x) = [\log(\sec x - \tan x)]^3$

$= \left(\log\left[\frac{\sec^2 x - \tan^2 x}{\sec x + \tan x}\right]\right)^3 = \left[\log\left(\frac{1}{\sec x + \tan x}\right)\right]^3$

$= -[\log(\sec x + \tan x)]^3$

$f(-x) = -f(x) \Rightarrow$ Odd function

For one-one function

$\boxed{\text{Using T-8(ii)}}$ $f'(x) = 3[\log(\sec x + \tan x)]^2 \times \frac{1 \cdot \sec x(\sec x + \tan x)}{(\sec x + \tan x)}$

$\Rightarrow f'(x) = 3[\log(\sec x + \tan x)]^3$ $\quad \underbrace{\sec x}_{>0 \text{ for } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)}$

Since, $f'(x) > 0$

$\Rightarrow$ One-one function

Since, range = codomain $\in R$. Hence, onto function.

22. **(c)** Put $x = -7$

$\Rightarrow f(-7) = -(a \cdot 7^7 + b \cdot 7^3 + c \cdot 7) - 5 \Rightarrow (a \cdot 7^7 + b \cdot 7^3 + c \cdot 7) = -12$

Now, put $x = 7$

$f(7) = \underbrace{a \cdot 7^7 + b \cdot 7^3 + c \cdot 7} - 5 = -12 - 5 = -17.$

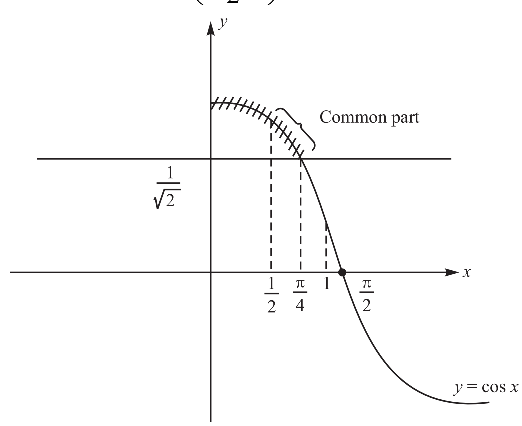
23. **(c)** $\boxed{\text{Using T-13}}$ Since period of $\cos x$ is 2π

$\Rightarrow$ Period of $f(x)$ is $\frac{2\pi}{3}$.

24. $x \in \left(\frac{1}{2}, \frac{\pi}{4}\right]$

(1) $\cos x - \frac{1}{\sqrt{2}} \geq 0$ $\qquad$ (2) $\frac{3}{2}x - x^2 - \frac{1}{2} > 0$

$\left(x - \frac{1}{2}\right)(x - 1) < 0 \qquad x \in \left(+\frac{1}{2}, 1\right)$



Hence, the common part is the graph will give the domain of $f(x)$

So the domain is $\quad x \in \left(\frac{1}{2}, \frac{\pi}{4}\right]$.

25. **(a)** $\boxed{\text{Using T-8(ii)}}$ $f'(x) = \frac{(1+x) - x}{(1+x)^2} > 0$

Hence $f(x)$ is one-one function

Since, in the co-domain $\rightarrow [0, \infty)$; '0' is included

But in domain $\quad x \neq 0 \Rightarrow f(x) \neq 0$

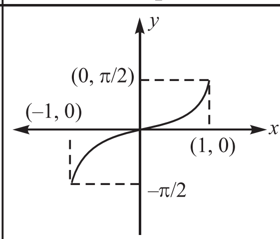
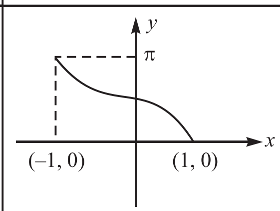
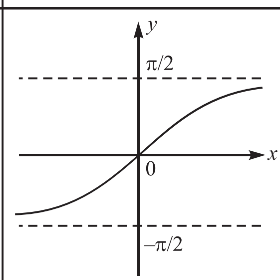
Hence, $\quad$ Range $\neq$ Co-domain $\Rightarrow$ Not onto

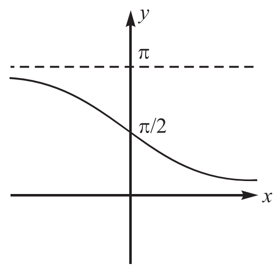
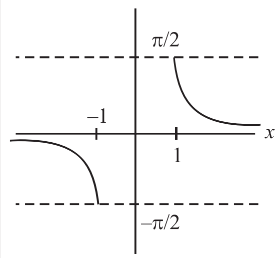
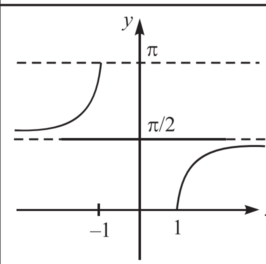
2 Inverse Trigonometric Functions

Review of Key Notes and Formulae

1. **Inverse Trigonometric Function:** As we know that trigonometric functions are not one-one and onto i.e. their natural domain and range, so their inverse do not exist but if we restrict their domain and range, then their inverse may exist.

2. **Graphs of Inverse Trigonometric Functions**

<table>
<tr><th></th><th>Function</th><th>Graph</th><th>Domain</th><th>Range</th></tr>
<tr><td>(i)</td><td>$y = \sin^{-1} x$</td><td>
</td><td>$x \in [-1, 1]$</td><td>$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$</td></tr>
<tr><td>(ii)</td><td>$y = \cos^{-1} x$</td><td>
</td><td>$x \in [-1, 1]$</td><td>$x \in [0, \pi]$</td></tr>
<tr><td>(iii)</td><td>$y = \tan^{-1} x$</td><td>
</td><td>$x \in R$</td><td>$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$</td></tr>
</table>

(iv)	$y = \cot^{-1} x$		$x \in R$	$y \in (0, \pi)$
(v)	$y = \text{cosec}^{-1} x$		$x \in (-\infty, -1] \cup [1, \infty)$	$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$
(vi)	$y = \sec^{-1} x$		$x \in (-\infty, -1] \cup [1, \infty)$	$y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

1. Properties of ITF (Inverse Trigonometric Function)

Property-1

(i) $\sin(\sin^{-1} x) = x$; $x \in [-1, 1]$
(ii) $\cos(\cos^{-1} x) = x$; $x \in [-1, 1]$
(iii) $\tan(\tan^{-1} x) = x$; $x \in \mathrm{R}$
(iv) $\cot(\cot^{-1} x) = x$; $x \in \mathrm{R}$
(v) $\text{cosec}(\text{cosec}^{-1} x) = x$; $x \in (-\infty, -1] \cup [1, \infty)$
(vi) $\sec(\sec^{-1} x) = x$; $x \in (-\infty, -1] \cup [1, \infty)$

Property-2

(i) $\sin^{-1}(\sin x) = x$; $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(ii) $\cos^{-1}(\cos x) = x$; $x \in [0, \pi]$
(iii) $\tan^{-1}(\tan x) = x$; $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(iv) $\cot^{-1}(\cot x) = x$; $x \in (0, \pi)$

(v) $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x\ ;\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(vi) $\sec^{-1}(\sec x) = x\ ; x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

Very Important Note:

If 'x' is not given according to above domain then make it between the above domain by using "$\pm n\pi$"

Ex:

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right)$$
$$= \sin^{-1}\left(\sin\frac{\pi}{3}\right)$$
$$= \frac{\pi}{3}.$$

Property-3

(i) $\sin^{-1}(-x) = -\sin^{-1} x \quad \forall\, x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x \quad \forall\, x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x \quad \forall\, x \in R$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x \quad \forall\, x \in R$

(v) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x \quad \forall\, x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1} x \quad \forall\, x \in (-\infty, -1] \cup [1, \infty)$

Property-4

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\ ;\ x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\ ;\ x \in R$

(iii) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}\ ;\ x \in (-\infty, -1] \cup [1, \infty)$

Property-5

(i) $\sin^{-1} x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right); x \in [-1, 1] - \{0\}$

(ii) $\cos^{-1} x = \sec^{-1}\left(\frac{1}{x}\right); x \in [-1, 1] - \{0\}$

(iii) $\tan^{-1} x = \begin{cases} \cot^{-1}\frac{1}{x} & ; x > 0 \\ -\pi + \cot^{-1}\frac{1}{x} & ; x < 0 \end{cases}$

Property-6

(i) $$\tan^{-1}x+\tan^{-1}y=\begin{cases}-\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x<0,\ y<0,\ xy>1\\ \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) ; & xy<1\\ \pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) ; & x>0,\ y>0,\ xy>1\end{cases}$$

(ii) $$\tan^{-1}x-\tan^{-1}y=\begin{cases}-\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right); & x<0,\ y<0,\ xy<-1\\ \tan^{-1}\left(\dfrac{x-y}{1+xy}\right) ; & xy>-1\\ \pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) ; & x>0,\ y<0 \text{ and } xy>-1\end{cases}$$

Property-7

$$\sin^{-1}x\pm\sin^{-1}y=\begin{cases}\sin^{-1}\left[x\sqrt{1-y^2}\pm\sqrt{1-x^2}\right]; x,\ y\ge 0 \text{ and } x^2+y^2\le 1\\ \pi-\sin^{-1}\left[x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right]; x,\ y\ge 0 \text{ and } x^2+y^2>1\end{cases}$$

Property-7.1

$$\cos^{-1}x\pm\cos^{-1}y=\begin{cases}\cos^{-1}\left[xy\mp\sqrt{1-x^2}\ \sqrt{1-y^2}\right]; x,\ y>0 \text{ and } x^2+y^2\le 1\\ \pi-\cos^{-1}\left[xy\mp\sqrt{1-x^2}\ \sqrt{1-y^2}\right]; x,\ y>0 \text{ and } x^2+y^2>1\end{cases}$$

Property-8: Simplified Trigonometric Functions

(i) $$\sin^{-1}\frac{2x}{1+x^2}=\begin{cases}-\pi-2\tan^{-1}x; & x<-1\\ 2\tan^{-1}x; & -1\le x\le 1\\ \pi-2\tan^{-1}x; & x>1\end{cases}$$

(ii) $$\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\begin{cases}\pi+2\tan^{-1}x ; & x<-1\\ 2\tan^{-1}x; & -1<x<1\\ -\pi+2\tan^{-1}x; & x>1\end{cases}$$

(iii) $$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\begin{cases}-2\tan^{-1}x ; & x\le 0\\ 2\tan^{-1}x ; & x\ge 0\end{cases}$$

(iv) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\dfrac{x+y+z-xyz}{1-(xy+yz+zx)}\right)$;

if $x>0, y>0, z>0$ and $(xy+yz+zx)<1$

★**Note:** (i) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$, then $xy + yz + zx = 1$

(ii) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then $xy + yz + zx = xyz$.

Important Points to Remember

(i) $\tan^{-1}\left(\frac{x}{\sqrt{a^2 - x^2}}\right) = \sin^{-1}\frac{x}{a}$

(ii) $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$

(iii) $\tan^{-1}\left(\frac{3a^2x - x^3}{a(a^2 - 3x^2)}\right) = 3\tan^{-1}\frac{x}{a}$

(iv) $\tan^{-1}x_1 + \tan^{-1}x_2 + \ldots\ldots + \tan^{-1}x_n = \tan^{-1}\left[\frac{S_1 - S_3 + S_5 \ldots\ldots}{1 - S_2 + S_4 - S_6 + \ldots\ldots}\right]$

where, S_k denotes the sum of the product of $x_1, x_2, \ldots, x_n$ taken 'K' at a time.

★**Note:** $\tan^{-1}(\sqrt{2}+1) = \frac{3\pi}{8}$

$\tan^{-1}(2 - \sqrt{3}) = \frac{\pi}{12}$

TIPS & TRICKS!

TIPS AND TRICKS: (T-1)

" '0' (zero) or '1' and you are done".

This is a substitution method such that we can substitute the values of x and y be '0' or '1' or vice-versa. This works in maximum cases. In case of any failure check for the other values of the variables (x, y, etc) like 1, 2, 3, ... etc.

★★**Note:** Always back check in this type.

Illustration 1

If $x > 0$, $y > 0$ and $x > y$, then find $\tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{x+y}{x-y}\right)$.

[AIEEE 2005]

(a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) $\frac{\pi}{2}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ Put $x = 1,\ y = 1$

$$\Rightarrow \quad \tan^{-1}1 + \tan^{-1}(\infty) = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$ **Ans. (a)**

Illustration 2

If $\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$, then $4x^2 - 4xy \cos \alpha + y^2 =$

(a) $-4 \sin^2 \alpha$ (b) $4 \sin^2 \alpha$ (c) 4 (d) $2 \sin 2\alpha$

Short-cut solution :

$\boxed{\text{Using T-1}}$ Put $x = y = 1 \Rightarrow \alpha = -\frac{\pi}{3}$

Now, put 'α' in the required eqn.

$$\Rightarrow 4x^2 - 4xy \cos\left(-\frac{\pi}{3}\right) + y^2 \qquad (\because x = y = 1)$$

$$\Rightarrow \quad 4 - 4 \times \frac{1}{2} + 1 = 3$$

Now, check option A, B, C, D for $\alpha = -\frac{\pi}{3}$

$$\Rightarrow \quad 4 \times \sin^2 \alpha = 4 \times \sin^2\left(-\frac{\pi}{3}\right) = 3$$ **Ans (b)**

Illustration 3

$$\tan^{-1}\left(\frac{a_1 x - y}{a_1 y + x}\right) + \tan^{-1}\left(\frac{a_2 - a_1}{1 + a_1 a_2}\right) + \tan^{-1}\left(\frac{a_3 - a_2}{1 + a_2 a_3}\right) + \ldots$$

$$+ \ldots \tan^{-1}\left(\frac{a_n - a_{n+1}}{1 + a_n a_{n-1}}\right) + \tan^{-1}\left(\frac{1}{a_n}\right).$$

Short-cut solution :

$\boxed{\text{Using T-1}}$ Put $a_1 = a_2 = a_3 = \ldots a_n = 0$

$$\Rightarrow -\tan^{-1}\left(\frac{y}{x}\right) + 0 + \ldots + 0 + \underbrace{\tan^{-1}(\infty)}_{\pi/2}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{y}{x} = \cot^{-1}\left(\frac{y}{x}\right).$$

Illustration 4

If $x > 0, y > 0$, then $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right) =$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$ or $-\frac{3\pi}{4}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ Put $x = y = 1$

$\Rightarrow \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$. **Ans. (b)**

Illustration 5

If $0 \le x \le \frac{1}{2}$, then value of $\tan\left(\sin^{-1}\left(\frac{x}{\sqrt{2}} + \frac{\sqrt{1-x^2}}{\sqrt{2}}\right) - \sin^{-1} x\right)$ is

(a) 0 (b) – 1 (c) 1 (d) $\frac{\pi}{4}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ Since $0 \le x \le \frac{1}{2}$, put $x = 0$

$$\Rightarrow \tan\left(\sin^{-1}\frac{1}{\sqrt{2}}\right) = \tan\frac{\pi}{4} = 1.$$ **Ans. (c)**

Illustration 6

If $x^2 + y^2 + z^2 = k^2$, then value of

$\tan^{-1}\left(\frac{xy}{zk}\right) + \tan^{-1}\left(\frac{xz}{yk}\right) + \tan^{-1}\left(\frac{zy}{xk}\right)$ is equals to

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 0

Short-cut solution :

$\boxed{\text{Using T-1}}$ Put $x = y = z = 1$

$\Rightarrow K^2 = 3 \Rightarrow K = \sqrt{3}$

Hence, $\tan^{-1}\left(\frac{1\cdot 1}{1\cdot\sqrt{3}}\right) + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{2}$. **Ans. (a)**

Illustration 7

The value of $\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\left(\frac{x}{y}\right)\right\}+\tan^{-1}\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\left(\frac{x}{y}\right)\right\}$

is equal to

(a) $\frac{x}{y}$ (b) $\frac{y}{x}$ (c) $\frac{2x}{y}$ (d) $\frac{2y}{x}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ Put $x = 1$ and $y = 2$

$$\Rightarrow \tan\left\{\frac{\pi}{4}+\frac{1}{2}\times\frac{\pi}{3}\right\}+\tan\left\{\frac{\pi}{4}-\frac{\pi}{3}\times\frac{1}{2}\right\} = \tan\left\{\frac{5\pi}{12}\right\}+\tan\left\{\frac{\pi}{12}\right\}$$

$$= \tan 75° + \tan 15° = 2+\sqrt{3}+2-\sqrt{3} = 4$$

Now, check for $x = 1$ and $y = 2$ in options (a), (b), (c), (d).

$\Rightarrow \frac{2y}{x} = 4.$ **Ans. (d)**

TIPS AND TRICKS: (T-2)

Short trick to solve series produces in inverse trigonometric functions.
In this we can take value of 'n' be 1, 2 or 3, ... etc. to minimize the steps. And after that check the option for those value of 'n' which has taken.

Illustration 8

The value of

$$\tan^{-1}\left(\frac{x}{1+2x^2}\right)+\tan^{-1}\left(\frac{x}{1+6x^2}\right)+\tan^{-1}\left(\frac{x}{1+12x^2}\right)+\ldots+n \text{ terms}$$

(a) $\tan^{-1}(n+1)x-\tan^{-1}x$ (b) $\tan^{-1}(n+1)x+\tan^{-1}x$
(c) $\tan^{-1}(n-1)-\tan^{-1}x$ (d) $\tan^{-1}(n-1)x-\tan^{-1}x$

Short-cut solution :

$\boxed{\text{Using T-2}}$ Put $n = 1$

$$\Rightarrow \tan^{-1}\left(\frac{x}{1+2x\times x}\right)=\tan^{-1}\left(\frac{2x-x}{1+2x\times x}\right)$$

As we know that $\tan^{-1}\left(\frac{A-B}{1+AB}\right)=\tan^{-1}A-\tan^{-1}B$

$\Rightarrow \tan^{-1}2x-\tan^{-1}x$

Now, check for $n = 1$ in options (a), (b), (c), (d). **Ans. (a)**

Illustration 9

The value of

$$\tan^{-1}\frac{1}{x^2+x+1}+\tan^{-1}\frac{1}{x^2+3x+3}+\tan^{-1}\frac{1}{x^2+5x+7}+\ldots n \text{ terms}$$

(a) $\tan^{-1}(x-n)-\tan^{-1}x$ (b) $\tan^{-1}(x+n)-\tan^{-1}x$

(c) $\tan^{-1}(n-x)-\tan^{-1}x$ (d) $\tan^{-1}(x+n+1)-\tan^{-1}x$

Short-cut solution :

$\boxed{\text{Using T-2}}$ Put $n = 1$

$$\Rightarrow \tan^{-1}\frac{1}{1+x(x+1)}=\tan^{-1}\left(\frac{(x+1)-1}{1+x(x+1)}\right)$$

As we know that $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$\Rightarrow \tan^{-1}(x+1)-\tan^{-1}1$

Check options (a), (b), (c), (d) for $n = 1$. **Ans. (b)**

TIPS AND TRICKS: (T-3)

In this trick first of all we will use substitution method as mentioned in T-1 and then stabilizing the components (options).

Illustration 10

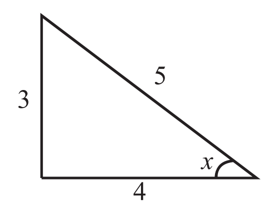
The value of $\cos^{-1}\left(\frac{4+5\cos x}{5+4\cos x}\right)$.

(a) $2\tan^{-1}\left(\frac{3}{4}\tan\frac{x}{2}\right)$ (b) $2\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)$

(c) $2\tan^{-1}\left(\frac{3}{5}\tan x\right)$ (d) $2\tan^{-1}\left(\frac{3}{4}\tan x\right)$

Short-cut solution :

$\boxed{\text{Using T-3}}$ Put $x = 90°$



$$\Rightarrow \cos^{-1}\left(\frac{4+0}{5+0}\right)=\tan^{-1}\left(\frac{3}{4}\right)$$

Now, we will check (a), (b), (c), (d) options.

Stabilizing $\Rightarrow$ $\tan^{-1}\frac{3}{4}=2\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)$

$\Rightarrow$ Take 'tan' to both sides $\Rightarrow$ $\frac{3}{4}=\frac{2\times\frac{1}{3}}{1-\frac{1}{9}}=\frac{3}{4}$.

Ans. (b)

Illustration 11

If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \dfrac{3\pi}{2}$ then,

(a) $x^{100} + y^{100} + z^{100} - \dfrac{9}{x^{101} + y^{101} + z^{101}} = 0$

(b) $x^{22} + y^{42} + z^{62} - x^{220} - y^{420} - z^{620} = 0$

(c) $x^{50} + y^{25} + z^{5} = 0$

(d) $\dfrac{x^{2008} + y^{2008} + z^{2008}}{(xyz)^{2009}} = 0$

Short-cut solution :

$\boxed{\text{Using T-3}}$ Put $\sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \dfrac{\pi}{2}$

$\Rightarrow x = y = z = 1$

Now, check options (a), (b), (c), (d) for $x = y = z = 1$ **Ans. (a, b)**

Illustration 12

Solve $\cos^{-1} x\sqrt{3} + \cos^{-1} x = \dfrac{\pi}{2}$, then $x =$

(a) $-\dfrac{1}{2}$ (b) $\dfrac{1}{2}$ (c) 0 (d) 1

Short-cut solution :

$\boxed{\text{Using T-1}}$

Check options (a), (b), (c), (d).

Put $x = \dfrac{1}{2} \Rightarrow \cos^{-1}\dfrac{\sqrt{3}}{2} + \cos^{-1}\dfrac{1}{2} = \dfrac{\pi}{6} + \dfrac{\pi}{3} = \dfrac{\pi}{2}$ **Ans. (b)**

Illustration 13

If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \dfrac{3\pi}{4}$ then the value of $xy + yz + zx$ is,

(a) 3 (b) 2 (c) -3 (d) -2

Short-cut solution :

$\boxed{\text{Using T-3}}$ Put $x = y = z = 1$

$\Rightarrow \tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \dfrac{3\pi}{4}$

Then, $xy + yz + zx = 3$ **Ans. (a)**

Illustration 14

$$\tan^{-1}\left(\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right);\ \frac{\pi}{2}\le x<\pi \text{ is equal to}$$

(a) $\frac{x}{2}-\frac{\pi}{2}$ (b) $\frac{\pi}{2}-\frac{x}{2}$ (c) $\pi - x$ (d) $2\pi - x$

Short-cut solution :

Using T-1 Put $x = \frac{\pi}{2}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{0+\sqrt{2}}{0-\sqrt{2}}\right) = -\tan^{-1} 1 = \frac{-\pi}{4}$$

Now, check options (a), (b), (c), (d)

$\Rightarrow \left(\frac{x}{2}-\frac{\pi}{2}=\frac{-\pi}{4}\right)$. **Ans. (a)**

Illustration 15

$$\left[\frac{1}{y^2}\left(\frac{\cos(\tan^{-1} y)+y\sin(\tan^{-1} y)}{\cot(\sin^{-1} y)+\tan(\sin^{-1} y)}\right)^2+y^4\right]^{\frac{1}{2}}$$ is equal to: **[JEE M 2013]**

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) $\sqrt{2}$

Short-cut solution :

Using T-1 Put $y = 1$

$\Rightarrow \quad [0^2 + 1] = 1$ **Ans. (a)**

SHORTCUTS: (SC-1)

In order to solve inequilities in inverse trigonometric functions use graphs to minimizing the steps.

Illustration 16

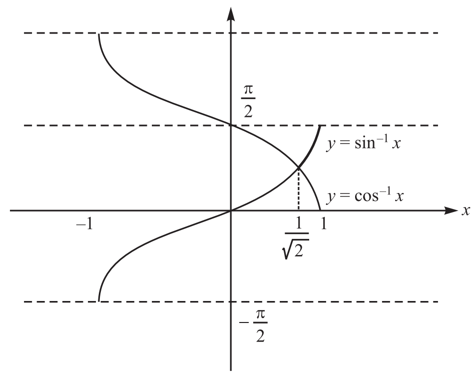
The values of 'x' for which $\sin^{-1} x > \cos^{-1} x \ \forall\ x \in (-1, 1)$.

Short-cut solution :

Using SC-1

We will draw graphs of both functions

$y = \sin^{-1} x$ and $y = \cos^{-1} x$



We will choose that part in the graph which is greater

$(\sin^{-1} x > \cos^{-1} x)$

Hence, it is clear from the graph $\Rightarrow\ x \in \left[\frac{1}{\sqrt{2}}, 1\right)$

TECHNIQUE

If the input of inverse trigonometric identity are infinite series and different then change the inputs in simple form using G.P. and then equate.

Illustration 17

If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - ....\right) = \frac{\pi}{2}$,

for $0 < |x| < \sqrt{2}$ then find the value of x.

Short-cut solution :

$\boxed{\text{Using Tech.}}$ $\because\ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... = \frac{x}{1 + \frac{x}{2}} = \frac{2x}{2 + x}$

And $x^2 - \frac{x^4}{2} + \frac{x^6}{4} - .... = \frac{x^2}{1 + \frac{x^2}{2}} = \frac{2x^2}{2 + x^2}$

$\therefore\ \sin^{-1}\left(\frac{2x}{2+x}\right) + \cos^{-1}\left(\frac{2x^2}{2+x^2}\right) = \frac{\pi}{2}$ and this is true when

$\frac{2x}{2+x} = \frac{2x^2}{2+x^2} \Rightarrow 2x + x^3 = 2x^2 + x^3$

$\Rightarrow\ 2x^2 - 2x = 0 \Rightarrow x(x-1) = 0 \quad \Rightarrow\ x = 0$ or $x = 1$

$\Rightarrow\ x = 1$ (x cannot be 0 as $0 < |x| < \sqrt{2}$)

Concept Booster Exercise

1. If cos tan^{-1} sin cot^{-1} x = P ; then find 'P'

(a) $\sqrt{\dfrac{x^2+1}{x^2+2}}$ (b) $\sqrt{\dfrac{x^2-1}{x^2+2}}$

(c) $\sqrt{\dfrac{x^2+1}{x^2-2}}$ (d) $\sqrt{\dfrac{x^2+3}{x^2-1}}$

2. $2\tan^{-1} x = \cos^{-1}\left(\dfrac{1-a^2}{1+a^2}\right) - \cos^{-1}\left(\dfrac{1-b^2}{1+b^2}\right)$; $(a > 0, b > 0)$

(a) $\dfrac{a+b}{1+ab}$ (b) $\dfrac{a+b}{1-ab}$ (c) $\dfrac{a-b}{1+ab}$ (d) $\dfrac{a-b}{1-ab}$

3. If $u = \cot^{-1}\sqrt{\cos 2\theta} - \tan^{-1}\sqrt{\cos 2\theta}$, then

(a) $1 + \sin u = \tan\theta$ (b) $1 + \sin u = \tan^2\theta$

(c) $1 - \sin u = \tan^2\theta$ (d) $\sin u = \tan^2\theta$

4. $\tan(\tan^{-1} x + \tan^{-1} y + \tan^{-1} z)$ is equal to:

(a) $1 + \cot(\cot^{-1} x + \cot^{-1} y + \cot^{-1} z)$

(b) $1 - \cot(\cot^{-1} x + \cot^{-1} y + \cot^{-1} z)$

(c) $\cot(\cot^{-1} x + \cot^{-1} y + \cot^{-1} z)$

(d) $\tan(1 + \tan^{-1} x + \tan^{-1} y + \tan^{-1} z)$

5. The value of $\sqrt{1+x^2}\,[\{x\cos(\cot^{-1} x) + \sin(\cot^{-1} x)\}^2 - 1]^{1/2}$ is equal to **[AIEEE 2008]**

(a) $\dfrac{x}{\sqrt{1+x^2}}$ (b) x (c) $x\sqrt{1+x^2}$ (d) $\dfrac{\sqrt{1+x^2}}{1+x^2}$

6. Let $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ where $|x| < \dfrac{1}{\sqrt{3}}$. Then the value of y is: **[JEE M 2015]**

(a) $\dfrac{3x-x^3}{1+3x^2}$ (b) $\dfrac{3x+x^3}{1+3x^2}$ (c) $\dfrac{3x-x^3}{1-3x^2}$ (d) $\dfrac{3x+x^3}{1-3x^2}$

7. Find value of $\sin^{-1}\cos(\sin^{-1}x) + \cos^{-1}\sin(\cos^{-1}x)$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{12}$

8. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then $\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}$ is equal to

(a) xyz (b) $\frac{1}{xyz}$ (c) 1 (d) 0

9. $\tan\left[\tan^{-1}\left(\frac{2}{1+1\cdot 2}\right) + \tan^{-1}\left(\frac{2}{1+2\cdot 3}\right) + \ldots + \tan^{-1}\left(\frac{2}{1+n(n+1)}\right)\right]$

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $-\frac{3}{2}$ (d) $-\frac{2}{3}$

10. If p, q, r are positive real numbers and

$$\theta = \tan^{-1}\left(\sqrt{\frac{p(p+q+r)}{qr}}\right) + \tan^{-1}\left(\sqrt{\frac{q(p+q+r)}{pr}}\right) + \tan^{-1}\left(\sqrt{\frac{r(p+q+r)}{pq}}\right),$$

then $\tan\theta$ is equal to

(a) 1 (b) 0 (c) $\frac{p+q+r}{pqr}$ (d) $\frac{pqr}{p+q+r}$

11. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then $x + y + z =$

(a) $xy + yz + zx$ (b) xyz (c) $\frac{1}{xyz}$ (d) $-xyz$

12. Let $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$ then

$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} =$$

(a) xyz (b) $\frac{1}{2}xyz$ (c) $2xyz$ (d) $\frac{xyz}{3}$

Numerical Value Problems

13. If $\cos^{-1}x + \cos^{-1}y = 2\pi$, and $\sin^{-1}x + \sin^{-1}y = P$, then find value of 4 + [P]; where [x] is greatest integer function.

14. Find number of solutions of $\cos^{-1}x = \tan^{-1}x\ \forall\ x \in [-1, 1]$.

15. Find number of solutions of $\sin^{-1}x = 1 - x\ \forall\ x \in [-1, 1]$.

16. The values of 'x' satisfying the inequality $\cos^{-1}x > \sin^{-1}x\ \forall\ x \in [-1, 1]$ is [–1, P], then find the value of [P]; where [ ] is greatest integer function.

17. The value of 'x' satisfying the inequality $2(\tan^{-1} x)^2 - 5\tan^{-1} x + 3 < 0$ is $(\tan K_1, \tan K_2)$ then find the value of $K_1 + \underset{(Two)}{\underset{\uparrow}{2}}K_2$.

Solutions

1. **(a)** $\boxed{\text{Using T-1}}$ Put $x = 0$

$$\Rightarrow \cos\tan^{-1}\sin\cot^{-1} 0 = \frac{1}{\sqrt{2}} = P$$

Now check options (a), (b), (c), (d) for $x = 0$

2. **(c)** $\boxed{\text{Using T-3}}$ Put $a = b = 1$

$\Rightarrow 2\tan^{-1} x = 0 \Rightarrow x = 0$

Now check options (a), (b), (c), (d) for $a = b = 1$

3. **(d)** $\boxed{\text{Using T-1}}$ Put $\theta = 0 \Rightarrow u = 0$

Now check options (a), (b), (c), (d) for $\theta = 0$ and $u = 0$

4. **(c)** $\boxed{\text{Using T-3}}$ Put $x = y = z = 0 \Rightarrow \tan 0 = 0$

Now check options (a), (b), (c), (d) for $x = y = z = 0$

5. **(c)** $\boxed{\text{Using T-1}}$ Put $x = 1$

$$\Rightarrow \sqrt{2}\left[\left\{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right\}^2 - 1\right]^{\frac{1}{2}} = \sqrt{2}$$

Now, check options a, b, c, d for $x = 1$

6. **(c)** $\boxed{\text{Using T-1}}$ Put $x = 1 \Rightarrow \tan^{-1} y = \frac{3\pi}{4} \Rightarrow y = -1$

Now, check options (a), (b), (c), (d) for $x = 1$

7. **(a)** $\boxed{\text{Using T-1}}$ Put $x = 0$

$$\Rightarrow \sin^{-1}\cos(0) + \cos^{-1}\sin\frac{\pi}{2} = \frac{\pi}{2}.$$

8. **(c)** $\boxed{\text{Using T-3}}$ Put, $\tan^{-1} x = \tan^{-1} y = \tan^{-1} z = \frac{\pi}{3}$

$\Rightarrow x = y = z = \sqrt{3}$

Now, $$\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

9. **(b)** Using T-2 Put $n = 1$

$\Rightarrow \tan\left(\tan^{-1}\left(\frac{2}{3}\right)\right) = \frac{2}{3}$.

10. **(b)** Using T-2 Put $p = q = r = 1$

$\Rightarrow \theta = \tan^{-1}\sqrt{3} + \tan^{-1}\sqrt{3} + \tan^{-1}\sqrt{3} = \pi$.

11. **(b)** Using T-3 Put $x = y = z = \sqrt{3}$

$\Rightarrow x + y + z = 3\sqrt{3} = xyz.$

12. **(c)** Using T-3 Put $\sin^{-1}x = \sin^{-1} y = \sin^{-1} z = \frac{\pi}{3}$

$\Rightarrow x = y = z = \frac{\sqrt{3}}{2}$

$\Rightarrow x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = \frac{3\times\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}$

Now, check options (a), (b), (c), (d) for $x = y = z = \frac{\sqrt{3}}{2}$

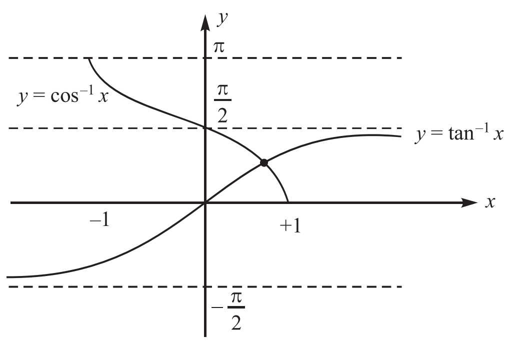
$\Rightarrow \quad 2\,xyz = 2\times\left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{4}$

13. **(0)** Using T-3 Put $\cos^{-1}x = \cos^{-1} y = \pi \quad \Rightarrow \quad x = y = -1$

Now, $\sin^{-1}(-1) + \sin^{-1}(-1) = P = -\pi$

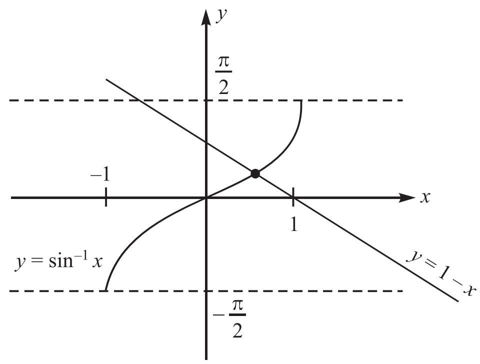
$\Rightarrow 4 + [-\pi] = 4 + (-4) = 0$

14. **(1)** Using SC-1 Drawing graphs of $y = \cos^{-1} x$ and $y = \tan^{-1} x$



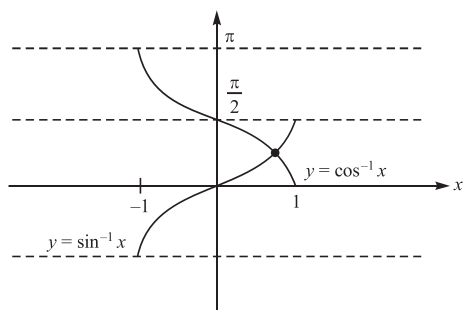
Number of intersection = 1 = No. of solution.

15. (1) $\boxed{\text{Using SC-1}}$ Drawing graphs of $y = \sin^{-1} x$ and $y = 1 - x$



Number of intersection = No. of solution = 1.

16. (0) $\boxed{\text{Using SC-1}}$ Drawing graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$



Number of intersection = No. of solution = 1.

$\therefore\ \cos^{-1}x > \sin^{-1}x$ when $x \in \left[-1, \dfrac{1}{\sqrt{2}}\right]$

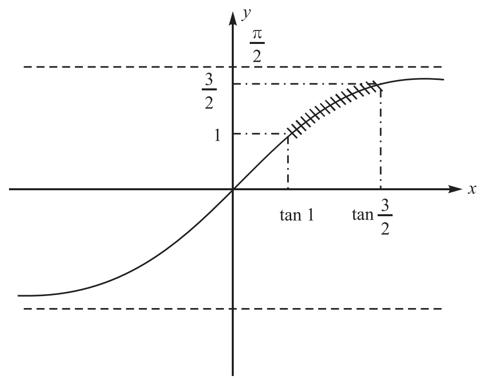
$\therefore\ P = \dfrac{1}{\sqrt{2}} \Rightarrow [\mathrm{P}] = \left[\dfrac{1}{\sqrt{2}}\right] = [0.707] = 0$

17. (4) $\boxed{\text{Using SC-1}}$ $2(\tan^{-1} x)^2 - 5\tan^{-1} x + 3 < 0$

$\Rightarrow\quad (\tan^{-1} x - 1)\left(\tan^{-1} x - \dfrac{3}{2}\right) < 0$

$\Rightarrow\quad \tan^{-1} x \in \left(1, \dfrac{3}{2}\right)$

Now, drawing graph of $y = \tan^{-1} x$



Hence, it is clear from graph that

$$x \in \left(\tan 1, \tan\frac{3}{2}\right) \Rightarrow K_1 = 1, K_2 = \frac{3}{2}.$$

Hence, $$K_1 + 2K_2 = 1 + \frac{3}{2}\cdot 2 = 4.$$