REVISED EDITION 2020

QUANTUM CAT

NEW CHAPTER
Number System

Specially Revised

Profit, Loss & Discount

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Geometry

Mensuration

2000 New MCQs

Also Useful for
XAT | SNAP | CMAT | MAT

Sarvesh K Verma

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Sarvesh K Verma
(An IIM Alumnus)

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QUANTUM CAT

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卐 **Administrative & Production Offices**

Regd. Office

'Ramchhaya' 4577/15, Agarwal Road, Darya Ganj, New Delhi -110002

Tele: 011- 47630600, 43518550; Fax: 011- 23280316

Head Office

Kalindi, TP Nagar, Meerut (UP) - 250002, Tel: 0121-7156203, 7156204

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PREFACE

Let me take this opportunity to introduce you to the revised edition of the most sought after book for Quantitative Aptitude.

At the very outset it would be vital to let you know that this edition covers all the essential concepts, related examples, handcrafted problems with multiple solutions which are pretty useful in cracking major competitive exams across the globe.

This book is designed keeping certain key aspects in the mind –

- Every topic starts with very basic level concepts so that anyone can easily strengthen one's foundation of that topic and one does not have to struggle for basic understanding.
- Every concept is discussed in detail and supported by the variety of examples. Some topics are supported by 40-50 examples with all sorts of twists in the situations and wordings. And that's how your mind starts thinking differently.
- Shortcuts have been devised in order to reduce the time to answer the problem. Since time is the main factor in competitive exams, so it is vital to think of those solutions that are fast, accurate and reliable.
- Most of the problems are solved using alternative methods. It gives you a 360- degree approach to understand the problem and apply your lateral thinking.
- Special alternative solutions and examples are there to help the students who have left mathematics long back due to probably no or little interest in mathematics.
- Self-explanatory answers enable the students to prepare for the toughest exams without joining the coaching centers, especially the ones which charges you a hefty price.
- This edition has got thousands of new problems based on latest patterns and difficulty level to keep you updated with the new dynamics of the competitive exams.
- And, this edition has got scores of new concepts included to make you stay ahead of your competitors. I am sure that you will never find many of these new concepts in any other book or study material until they copy the same from the Quantum CAT. And, that is precisely the reason behind revising this mammoth book so that you have an extra edge over your peers.
- Every topic has 2-3 levels of questions making you prepare for the most difficult problems even beyond the CAT level. Practise these exercises based on the difficulty level of your exam.

From my experience most of the students believe in and require a through practice .That's why the number of problems based on any topic is way beyond your imagination. If you are serious about your exam, start early and refer this book smartly. There is no wonder why some 30-40 percentilers have become 99+ percentilers in 3-4 months after solving this book. This book has helped many students secure 100 percentile in CAT, All India Rank 1 in RBI Grade B exam, SBI PO and SSC CGL.

We all know that most of the students are scared about Quantitative Aptitude section. It happens because when they see a lot of graphs, functions and algebraic equations and try to solve these problems in a traditional manner they feel puzzled. However, I suggest them to look at these problems differently if they want their tension to disappear. Even I personally do not use too many formulae or equations while solving the problem. I try to use option-elimination method, or substitution method or approximation method or diagrammatic approach or unit digit approach or some other method different from standard and traditional mathematics. After all, no one expects you to know all the theorems and formulae of mathematics. Examiners expect you to know that if you can look at the numbers, mathematical figures and graphs smartly and make some sense out of it. If you have a smart way to look at these things, you are naturally a problem solver. The idea is to test your common sense and not to identify a Ramanujan or Aryabhatta in a future employee. There are numerous such solutions that have been offered in this book to develop your thought process as expected by the smart examiners.

For example, in the following figure (i), find the area of the smaller equilateral triangle if the area of the larger equilateral triangle is $60\sqrt{3}$ units. More often than not you might struggle for 2 to 3 minutes to find the relevant data, geometrical theorems and appropriate formulae. However, if you look at the figure (ii) in which I have just rotated the smaller equilateral triangle without changing its shape or size, you can easily tell me that the area of smaller triangle inside the circle is 1/4th of the larger triangle outside the circle, as the vertices of the smaller equilateral triangle meet the mid-points of the sides of the larger equilateral triangle and thereby creating 4 same size equilateral triangles. That means the area of the inscribed triangle is $15\sqrt{3}$ units.

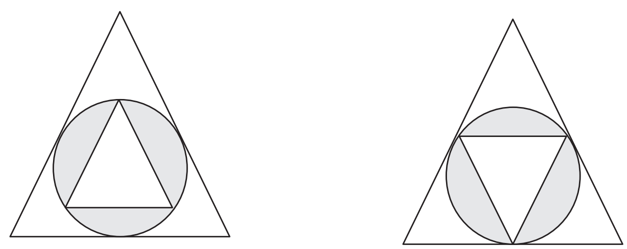

Fig (i) Fig (ii)

I request you to start and finish the chapter in the same order as they are given in the book. Also if you are not getting a desirable decent score, try solving each and every example and problem of the particular chapter. Mark and revise the difficult sums quite frequently and consistently to boost your confidence and score.

I have tried my best to keep it error free, however if any sort of error or discrepancy is found, it must be communicated to me if it is related to content, otherwise it must be communicated to Arihant publications and in case of any delivery issue it must be communicated to the delivery partner such as Amazon or Flipkart.

I am thankful to each one of you for letting me use this space to talk about the product – the reason for its creation, the reason for its success, the reason why one should trust it and the key features of the book, if one wants to understand the book before going through all the main pages. Now let me share some unusual yet important facts about this book with you.

- After regular appearance of its questions in CAT since its debut, from last couple of years its questions have started appearing in Bank PO and SSC CGL exams in conspicuously large numbers, as SBI PO and SSC CGL exams have increased their difficulty level a couple of notches up. Why do these exams choose to pick up the problems from Quantum CAT?

- It is because I strongly believe that only innovation driven products set the standards for others to take a leaf from. As far as competitive exams are concerned I try to design the problems that become unavoidable for a paper setter. And, that's why our students have always felt delighted to see the same or very similar questions of Quantum CAT appearing in their papers. Trust me no student want to face the uncertainty in the exam. And, that's why you could be able to experience the variety, depth and subtlety in the problems given in the Quantum CAT. Reinforcing the same philosophy, I have revamped the material of the book for the years to come.

> ***Hello Sir,*** *8 November 2012 at 12:54 PM*
>
> *I appeared for my CAT exam on 5th Nov and I have to tell you that around 12 sums, from Number System, Time & Work and Geometry were directly picked up from your book. 2 sums were exactly word to word from the book and I marked the answer even before solving them (Although I didn't get overconfident and solved them before marking). Thanks a tonne! :)) I can't tell u how happy I was to see questions that I had already done before!*
>
> *A big thanks again!*
>
> ***Swati Daga***

- I am fortunate enough to be able to write a book that has been inspiring its readers and teachers alike. It does not help any of us to bask in the past glory, however, sharing my perspective with deep sense of humility will definitely help you understand the point that I am putting across. As an esteemed reader of this book you must be proud of the fact that whenever I design the problems or introduce a new concept it is well thought out from the perspective of what could be the future of the exam you are preparing for. If I am not able to visualize and foresee the trend, I won't be able to do justice to my work. Precisely, that has been the reason that I never felt to revise the Quantum CAT in last 15 years, while the other books on Quantitative Aptitude had to go through multiple revisions. In a hyper competitive environment as a student you deserve nothing but the best. When even 0.25 marks can jettison you out of the cut off list, my responsibility becomes huge in the sense that how could I develop a product which is as reliable as your parents. And, I am sure that when you see the new set of questions in Quantum CAT and you compare them with any other resource in the market, you will definitely find the positive difference with Quantum CAT.

> ***Hello Sir,*** *May 24, 2015*
>
> *My name is Raajit and I will be joining the PGP at IIM Ahmedabad this year. I had scored an 88 percentile in my first attempt. For the second attempt, I used your quantitative aptitude book (Quantum CAT) for preparation and through a lot of mock tests, I scored a 99+ percentile. I just personally wanted to thank you for this and when I came across your profile on Quora, I could not help but send a thank you message.*
>
> ***Raajit Kumar***
> *IIM Ahmedabad, PGP 2015-17*

Hi Sir, *9 July 2015 at 2:48 AM*

My name is Krishnan, and I am currently based in London. I used your book – Quantum CAT extensively during 2012, and it helped me crack the CAT. I did not join the IIM at that point, but choose to continue working. Now, in 2015, I have decided to write the GMAT, and have ordered your book from India, and I am happy to return to this book after about 2 years. I would like to Congratulate you on coming up with such a wonderful book.

Thanks
Krishnan

In my opinion, a book is not just a collection of pages or random thoughts; rather it represents a personality, a belief and a purpose. I have tried my best to provide you a one-stop solution. However, I am always open to listen to your feedback and suggestions at my official email ID mentioned below. Please do get in touch with me with your valuable inputs. If you really love the book please do share your thoughts with your friends on social media. Also, you can use the hashtag #QuantumCAT while sharing your thoughts on social media.

lifetimeinnovations@yahoo.co.in

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www.facebook.com/I.am.Sarvesh.Kumar.Verma

I wish you all the best for your bright career!

Author

ACKNOWLEDGEMENT

Let me take this opportunity to express my deepest gratitude to each and every person and institution that has been pivotal in my journey of writing the Quantum CAT and making it an enormously successful book.

First and foremost, I am thankful to maa Saraswati, the Goddess of Knowledge. I have no words to describe my feelings full of gratitude towards my beloved parents and venerable teachers. Parents have done everything for me – every support, every care and every sacrifice one can imagine. I wish every child could get the kind of support I always get from my parents. Likewise, my teachers have invariably influenced me, encouraged me, pushed me, and guided me to excel in my life, no matter what. I extend my deep gratitude to my sensei, Dr. Daisaku Ikeda for helping me realize my true potential by expanding my life state. Without him I would not have accomplished the revised edition of Quantum CAT.

I would like to thank my students spread around the globe who have encouraged, inspired and challenged me to come up with a world-class product from India. Without their ideas, questions, doubts and aspirations to succeed I would not have been able to understand the nuances of writing a book at such an early age in my life.

I deeply acknowledge the support of Jai Bhan Verma, who was instrumental in motivating me to accept this offer and embark on this promising journey of writing a book for millions of Indians. I am equally indebted to my friend and well wisher Praveen Verma, who eventually had to support me in the proof reading apart from being the most supportive person since the very first day.

I have a deep sense of gratitude towards Ms. Anjuli Verma, my dearest sister, who has helped me immensely in developing the multiple choices for questions, proof reading the material, getting the courier ready to be shipped. She has worked equally hard since the very first day. If she would not have supported me, probably, I could not have done it alone.

I am highly indebted to Ms. Saundarya Katiyar, my cofounder, who has always been highly supportive, encouraging and of course she has been proof reading, contributing to a large pool of new questions and new ideas in the book for a couple of years. However, this year she had made immense contribution to the new edition of Quantum CAT, as preparing the revised edition has been a daunting and time consuming task. Without her support the revised edition would have never been with you at all.

My heartfelt gratitude to Mr. Abis Naqvi, Mr. S. A. Siddiqui, Mr. Bhanu Pratap Singh, Mr. Amza Rahman, Mr. Ravi Singh, Ms. Kiran Verma, Mr. (Col.) D. R. Patnaik, Pravesh K. Verma for their invaluable contribution in my book and in my life. Heartily, I have a deep regard for each one of you, as I owe my success to each one of you.

How can I forget to mention the contribution of all my friends, colleagues and well wishers who believed in my hard work and sincerity to accomplish a monumental book ! I'm grateful for having your back every moment of my life!

My deepest regard for those people who have shared their feedback and constructive criticism. Their efforts have paved the way and have helped me realize their invaluable contribution in enhancing the quality and accuracy of the content. You will always live in my heart. I request them if anyone wants to contribute to the Quantum CAT by sharing some ideas, questions, answers and doubts one can join the Quantum CAT Facebook Group: www.facebook.com/groups/117572951590692

The truth is how can a book see the light of the day without the strong commitment, patience and vision of Mr. Deepesh Jain and his dynamic team at Arihant publications. I would like to express my deep regard to Mr. Jain for bearing with me for so many years! And to my dearest friend Mr. Harsh Kumar, I am really touched by your patience, your interpersonal skills and for everything you have been doing for me since you joined the Publications. And, to all the teams from composing, typesetting, printing, logistics, marketing and accounting, all of you have been really a tremendous support to me. Without your skills and efforts this book would not have disrupted the market.

DEDICATION

To all the cosmic forces and deities who have blessed me with such a wonderful life in the human form with amazing parents, siblings extended family, friends, teachers, mentors, critics and well wishers!

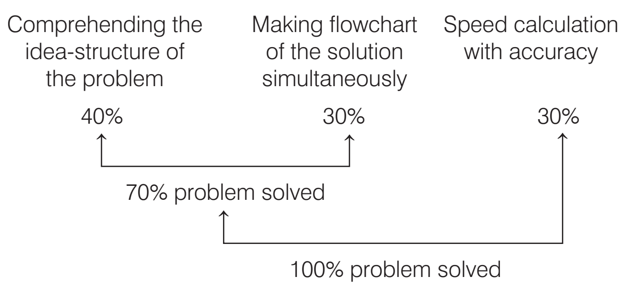
To the ones who made me realize how much my life is beautiful! I wish I could dedicate not just my book, but also my whole being till eternity. The fun does not lie in writing a book rather it lies in the dynamics of expression of life that make us infinite. Let's continue to cherish and share a smile with the ones who need this book more than anything else at one point of time!

CONTENTS

CHAPTER 01

Number System

First of all I would like to mention that this chapter is deliberately written to promote your interest for CAT preparation in a systematic approach which will lay the strong foundation. Since most of the problems (*i.e.,* 20-40%) in CAT paper (*viz.,* QA section) belong to this chapter alone hence it becomes inevitable to discuss the nuances and subtleties of the various concepts objectively and comprehensively. That's why this chapter has become very bulky too. In your own interest it would be better to adhere to my advices and stipulations stated at appropriate places in this chapter even throughout the book. As per my own experiences there are basically three stages in the solution of a problem and each of the stage has its own contribution. The following diagrams illustrates the importance of each stage.



If any of the three above mentioned activities is botched up then you are prone to failure. Hence, it is obvious that speed calculation invariably enhances your rate of success in CAT and most importantly it eliminates the stress and anxiety to ensconce you in your own comfort zone. Remember that you can succeed only when you operate in your comfort zone else it turns out to be a challenge. Due to above mentioned reasons it is imperative to become handy in quick calcualtion which is required in DI too. The latter part, even major section, of the chapter is devoted to numbers, their kinds, operations and behaviour in mathematical milieu.

You are supposed to not to skip a bit of it since every bit and every concept is equally important. It is advised that you should make the flow chart of the solution while reading, since going back to the problem several times means irritation and wastage of invaluable time. Also, you should try to solve maximum no. of problems without pencil and paper since it makes you a quick respondent and saves a lot of time. Finally, avoid rote method of learning in maths instead be inqusitive and explorer to gain an advantage. The bottomline is that perceive logic and apply logic, since it is logical.

Chapter Checklist

- *Calculation Techniques*
- *Square and Cubes*
- *Basic Numbers*
- *Integers*
- *Factors and Multiples*
- *HCF and LCM*
- *Fractions and Decimal Fractions*
- *Indices and Surds*
- *Factorial, Last Digits and Remainders*
- *Rational and Irrational Number*
- *Various Number Systems*
- *Potterns, Relations and Functions*
- ***CAT Test***

1.1 Calculation Techniques

Multiplication Rules

Multiplication is nothing but the shortest method of addition. In day-to-day life when we have to add up the same quantity many a time, so we multiply the given number by the number of times to which the given number is to be added repeatedly *e.g.*, $3+3+3+3+3+3+3$ can be added seven times but to get quick result we just multiply 3 by 7.

Hence, we get $3 \times 7 = 21$. Thus, I am sure that now you must have become aware of the fact that how much important is the multiplication table. So, I hope that you will learn the multiplication table.

Now, I would like to suggest you that in the competitive exams like CAT, where speed and accuracy are totally indispensable. So there are some novel and enticing techniques for multiplication given in this chapter. But in some techniques you have to know the square of some relevent numbers, normally the squares of 1 to 100. Last but not the least you don't need to bother about since I have some very innovative techniques to learn and calculate the squares very readily. Even most of the times you need not to use a pencil and paper for this work and that's my purpose.

Basically we divide these techniques into two categories. The first one which I have derived from some algebraic operations and the second one belongs to our ancient maths *i.e.*, Vedic maths. In this book I have laid emphasis on my own approach of multiplication, because years of research in numeric maths I have found that I myself and my students find it comfortable as it is very convenient and pragmatic to do within a few seconds.

Case 1. Multiplication by 5, 25 $(= 5^2)$, 125 $(= 5^3)$, 625 $(= 5^4)$ etc.

Exp. 1) Multiply 187 by 5 i.e., 187×5.

Solution **Step 1:** Associate as many zeros at the end of the given number as there is the power to 5. Thus we write 187 as $187\underline{0}$. (because $5 = 5^1$)

Step 2: Divide the resultant number by 2 as many times as there is the power to 5. Thus, $1870 / 2 = 935$

Exp. 2) Multiply 369 by 125.

Solution **Step 1:** 369000 ($\because 125 = 5^3$)

hence 3 zeros will be associated.

Step 2: $\dfrac{369000}{2 \times 2 \times 2} = \dfrac{369000}{8} = 46125$

(Since, there is 5^3 hence it will be divided by 2^3.)

In general, if you want to multiply any number K by 5^n, then you just need to divide that number by 2^n, after writing or (associating) 'n' zeros at the end of the number *i.e.*,

$$K \times 5^n = \frac{K0000 \ldots n \text{ zeros}}{2^n}$$

e.g., $732 \times 5^6 = \dfrac{732000000}{2 \times 2 \times 2 \times 2 \times 2 \times 2} = \dfrac{732000000}{64} = 11437500$

Case 2. Multiplication of a number by 9^n

i.e., $9, 9^2\ (= 81), 9^3\ (= 729)$ etc.

Exp. 1) Multiply 238 by 9.

Solution Multiply the given number by 10 and then subtract the given number from the resultant number. So

$$238 \times 9 = 238 \times (10 - 1) = 2380 - 238 = 2142$$

[using distributive law as $a \times (b - c) = ab - ac$]

Exp. 2) Multiply 238 by 81.

Solution
Step 1 : $238 \times 81 = 238 \times 9 \times 9$
Step 2 : $238 \times (10-1) \times (10-1)$
Step 3 : $[2380 - 238] \times (10 - 1)$
Step 4 : $2142 \times (10 - 1)$
Step 5 : $21420 - 2142 = 19278$

NOTE To multiply a number by the higher powers of 9 *i.e.*, 9 to power 3, 4, 5, ..., etc. It is better to do it conventionally.

Case 3. Multiplication of a number by the numbers whose unit (*i.e.*, last) digit is 9.

Exp. 1) Multiply 287 by 19.

Solution
$$287 \times (10 + 9) = 287 \times 10 + 287 \times 9$$
$$= 2870 + 287 \times (10 - 1)$$
$$= 2870 + 2870 - 287 = 5740 - 287 = 5453$$

or
$$287 \times 19 = 287 \times (20 - 1) = 287 \times 20 - 287$$
$$= 287 \times 2 \times 10 - 287$$
$$= 574 \times 10 - 287 = 5740 - 287 = 5453$$

Case 4. Multiplication of a number by the number which contains all the digits as 9.

Exp. 1) Multiply 743 by 99.

Solution $743 \times 99 = 743 \times (100 - 1) = 74300 - 743 = 73557$

Exp. 2) Multiply 23857 by 9999.

Solution
$$23857 \times 9999 = 23857 \times (10000 - 1)$$
$$= 238570000 - 23857 = 238546143$$

NOTE In general to multiply a number K by a number which is purely consists of n 9s, put n zeros at the end of the number K, then subtract the given number from the resultant number.

e.g., $K \times 9999 \ldots n \text{ times} = (K0000 \ldots n \text{ zeros}) - K$

Case 5. To multiply those two numbers, the sum of whose unit digit is 10 and the rest digits (left to the unit digit) are same in both the numbers.

Exp. 1) Multiply 38 by 32.

Solution

Step 1: Multiply the unit digits as $8 \times 2 = 16$

Step 2: Add 1 to the left digits, then multiply this increased number by the original digits *i.e.*, $3 \times (3 + 1) = 12$

Thus, we get $38 \times 32 = 1216$

Exp. 2) Multiply 83 by 87.

Solution **Step 1:** $3 \times 7 = 21$

Step 2: $8 \times (8+1) = 72$

$\therefore$ $83 \times 87 = 7221$

Exp. 3) Multiply 71 by 79.

Solution **Step 1:** $9 \times 1 = 09$

Step 2: $7 \times (7+1) = 56$

$\therefore$ $71 \times 79 = 5609$

NOTE The last two places are reserved for the product of unit digits as in the above example $1 \times 9 = 9$ but we fill the last two places by writing "09". So don't write 569 since it is wrong.

Exp. 4) Multiply 125 by 125.

Solution **Step 1:** $5 \times 5 = 25$

Step 2: $12 \times (12+1) = 156$

So, $125 \times 125 = 15625$

Exp. 5) Multiply 431 by 439.

Solution **Step 1:** $1 \times 9 = 09$

Step 2: $43 \times (43+1) = 1892$

So, $431 \times 439 = 189209$

Exp. 6) Multiply 1203 by 1207.

Solution **Step 1:** $3 \times 7 = 21$

Step 2: $120 \times (120+1) = 14520$

So, $1203 \times 1207 = 1452021$

Case 6. To multiply the two numbers whose difference is always an even number.

Exp. 1) Multiply 38 by 52.

Solution **Step 1:** $52 - 38 = 14$ (here 14 is an even number)

Step 2 : $\frac{14}{2} = 7$

Step 3 : $38 + 7$ or $52 - 7 = 45$

Step 4 : $(45)^2 = 2025$

Step 5 : $7^2 = 49$

Step 6 : $2025 - 49 = 1976$

Exp. 2) Multiply 76 by 96.

Solution **Step 1:** $96 - 76 = 20$ (here 20 is an even number)

Step 2: $\frac{20}{2} = 10$

Step 3: $76 + 10 = 96 - 10 = 86$

Step 4: $(86)^2 = 7396$

Step 5: $(10)^2 = 100$

Step 6: $7396 - 100 = 7296$

Exp. 3) Multiply 185 by 215.

Solution

Step 1: $215 - 185 = 30$ (here 30 is an even number)

Step 2: $\frac{30}{2} = 15$

Step 3: $185 + 15 = 215 - 15 = 200$

Step 4: $(200)^2 - (15)^2 = 40000 - 225 = 39775$

NOTE 1. In general, if there are any two numbers N_1 and N_2 and the difference between N_1 and N_2 is $2d$ then their product *i.e.*,

$$N_1 \times N_2 = (N_1 + d)^2 - d^2$$

or

$$N_1 \times N_2 = (N_2 - d)^2 - d^2$$

2. Even this method can be applied for the two numbers whose difference is an odd number but since it contains decimal values so some students may feel it slightly difficult.

Exp. 1) Multiply 17 by 22.

Solution **Step 1:** $22 - 17 = 5$

Step 2: $\frac{5}{2} = 2.5$

Step 3: $17 + 2.5 = 22 - 2.5 = 19.5$

Step 4: $(19.5)^2 - (2.5)^2 = 380.25 - 6.25 = 374.00$

Shortcut To multiply any two numbers say, N_1 and N_2 we apply the following methods.

$$N_1 \times N_2 = \left(\frac{N_1 + N_2}{2}\right)^2 - \left(\frac{N_2 - N_1}{2}\right)^2$$

So, if you know the square of the required numbers then using this method you can multiply any two numbers within 2-3 seconds as $56 \times 64 = \left(\frac{56+64}{2}\right)^2 - \left(\frac{64-56}{2}\right)^2$

$$= (60)^2 - (4)^2$$
$$= 3600 - 16$$
$$= 3584$$

Vedic Methods of Multiplication

Exp. 1) Solve 23×74.

Solution

$$\begin{array}{r} 23 \\ \underline{74} \\ 2 \end{array} \quad 4 \times 3 = 12$$

$$\begin{array}{r} 23 \\ \underline{74} \\ 02 \end{array} \quad (4 \times 2 + 7 \times 3) + 1 = 30$$

$$\begin{array}{r} 23 \\ \underline{\times 74} \\ \underline{1702} \end{array} \quad (7 \times 2) + 3 = 17$$

$$\begin{array}{r} 23 \\ \updownarrow \\ \underline{74} \\ 2 \end{array} \quad 12$$

$$\begin{array}{r} 2\ \ 3 \\ \times \\ \underline{7\ \ 4} \\ 02 \end{array} \quad 29 + 1 = 30$$

$$\begin{array}{r} 23 \\ \updownarrow \\ \underline{74} \\ 1702 \end{array} \quad 14 + 3$$

Exp. 2) Solve 83×65.

Solution

83 65 5	$5 \times 3 = 15$	83 65 5	15
83 ×65 95	$(5 \times 8 + 6 \times 3) + 1$ $= 59$	8 3 6 5 95	$58 + 1 = 59$
83 65 5395	$(6 \times 8) + 5$ $= 53$	83 65 5395	$48 + 5$

Exp. 3) Solve 524×19.

Solution

524 19 6	$9 \times 4 = 36$	524 19
524 ×19 56	$(9 \times 2 + 1 \times 4) + 3$ $= 25$	5 2 4 1 9
524 ×19 956	$(9 \times 5 + 1 \times 2) + 2$ $= 49$	5 2 4 1 9
524 ×19 9956	$(1 \times 5) + 4$ $= 9$	524 19

Exp. 4) Evaluate 3456×89.

Solution

3456 89 4	$9 \times 6 = 54$	3456 89
3456 89 84	$(9 \times 5 + 8 \times 6) + 5$ $= 98$	34 5 6 8 9
3456 89 584	$(9 \times 4 + 8 \times 5) + 9$ $= 85$	34 5 6 8 9
3456 89 7584	$(9 \times 3 + 8 \times 4) + 8$ $= 67$	3456 89
3456 89 307584	$8 \times 3 + 6$ $= 30$	3456 89

Exp. 5) Evaluate 245×367.

Solution

245 367 5	$5 \times 7 = 35$	245 367
245 367 15	$(7 \times 4 + 6 \times 5) + 3$ $= 61$	2 4 5 3 6 7
245 367 915	$(7 \times 2 + 6 \times 4 + 3 \times 5\ ) + 6$ $= 59$	2 4 5 3 6 7
245 367 9915	$(6 \times 2 + 3 \times 4) + 5$ $= 29$	2 4 5 3 6 7
245 367 89915	$(3 \times 2) + 2$ $= 8$	245 367

Exp. 6) Evaluate 123456×789.

Solution

123456 789 4	$9 \times 6 = 54$	123456 789
123456 789 84	$(9 \times 5 + 8 \times 6) + 5 = 98$	123456 789
123456 789 784	$(9 \times 4 + 8 \times 5 + 7 \times 6) + 9 = 127$	123456 789
123456 789 6784	$(9 \times 3 + 8 \times 4 + 7 \times 5) + 12 = 106$	123456 789
123456 789 06784	$(9 \times 2 + 8 \times 3 + 7 \times 4) + 10 = 80$	123456 789
123456 789 406784	$(9 \times 1 + 8 \times 2 + 7 \times 3) + 8 = 54$	123456 789
123456 789 7406784	$(8 \times 1 + 7 \times 2) + 5 = 27$	123456 789
123456 789 97406784	$(7 \times 1) + 2 = 9$	123456 789

Exp. 7) Find the value of 4567×1289.

Solution

$$\begin{array}{r} 4567 \\ \underline{1289} \\ \underline{3} \end{array} \quad 9 \times 7 = 63$$

$$\begin{array}{r} 4567 \\ \underline{1289} \\ \underline{63} \end{array} \quad (9 \times 6 + 8 \times 7) + 6 = 116$$

$$\begin{array}{r} 4567 \\ \underline{1289} \\ \underline{863} \end{array} \quad (9 \times 5 + 8 \times 6 + 2 \times 7) + 11 = 118$$

$$\begin{array}{r} 4567 \\ \underline{1289} \\ \underline{6863} \end{array} \quad (9 \times 4 + 8 \times 5 + 2 \times 6 + 1 \times 7) + 11 = 106$$

$$\begin{array}{r} 4567 \\ \underline{1289} \\ \underline{86863} \end{array} \quad (8 \times 4 + 2 \times 5 + 1 \times 6) + 10 = 58$$

$$\begin{array}{r} 4567 \\ \underline{\times 1289} \\ \underline{886863} \end{array} \quad (2 \times 4 + 1 \times 5) + 5 = 18$$

$$\begin{array}{r} 4567 \\ \underline{1289} \\ \underline{5886863} \end{array} \quad (4 \times 1) + 1 = 5$$

Exp. 8) Find the value of 325768×1234.

Solution

$$\begin{array}{r} 325768 \\ \underline{1234} \\ \underline{2} \end{array} \quad 4 \times 8 = 32$$

$$\begin{array}{r} 325768 \\ \underline{1234} \\ \underline{12} \end{array} \quad (4 \times 6 + 3 \times 8) + 3 = 51$$

$$\begin{array}{r} 325768 \\ \underline{1234} \\ \underline{712} \end{array} \quad (4 \times 7 + 3 \times 6 + 2 \times 8) + 5 = 67$$

$$\begin{array}{r} 325768 \\ \underline{1234} \\ \underline{7712} \end{array} \quad (4 \times 5 + 3 \times 7 + 2 \times 6 + 1 \times 8) + 6 = 67$$

$$\begin{array}{r} 325768 \\ \underline{1234} \\ \underline{97712} \end{array} \quad (4 \times 2 + 3 \times 5 + 2 \times 7 + 1 \times 6) + 6 = 49$$

$$\begin{array}{r} 325768 \\ \underline{1234} \\ \underline{997772} \end{array} \quad (4 \times 3 + 3 \times 2 + 2 \times 5 + 1 \times 7) + 4 = 39$$

$$\begin{array}{r} 325768 \\ \underline{1234} \\ \underline{1997712} \end{array} \quad (3 \times 3 + 2 \times 2 + 1 \times 5) + 3 = 21$$

$$\begin{array}{r} 325768 \\ \underline{1234} \\ \underline{01997712} \end{array} \quad (2 \times 3 + 1 \times 2) + 2 = 10$$

$$\begin{array}{r} 325768 \\ \underline{1234} \\ \underline{401997712} \end{array} \quad (1 \times 3) + 1 = 4$$

Divisibility Rules

While solving the mathematical problems we need to divide one number by another number. In general we use four terms for this process as

$$\text{Divisor}\,\overline{)\,\text{Dividend}\,}(\,\text{Quotient} \\ \underline{xyz} \\ \underline{\text{Remainder}}$$

or Dividend = (Divisor × Quotient) + Remainder

for example $30 = (7 \times 4) + 2$

where quotient is a whole number *i.e.*, 0, 1, 2, 3, … etc and remainder is always less than 'divisor' or zero.

So, when the remainder is zero, we can say that the given number is divisible by the particular divisor number.

$$\text{Now,} \quad 238 \div 6 = 39\frac{4}{6} = 39.66 \quad \text{(decimal notation)}$$

Since the above number when divided by 6 leaves a remainder of 4 then it is said to be not divisible by 6 where 238 is dividend, 39 is quotient obtained when divided by 6 as a divisor. Thus we get 4 as a remainder.

Further if we subtract the remainder from the dividend then the dividend (the resultant number) becomes divisible by particular divisor. *e.g.*,

$$238 - 4 = 234 \quad \text{(dividend–remainder)}$$

$$\begin{array}{r} 6\overline{)\,234\,}(39 \\ \underline{18} \\ 54 \\ \underline{54} \\ \times \end{array}$$

Here, the remainder is zero and the number is perfectly divisible by 6.

Another point is that if we add the difference of divisor and remainder to the given number (*i.e.*, dividend) the number becomes divisible by that particular divisor. *e.g.*,

$$6 - 4 = 2 \quad \text{(Divisor–Remainder)}$$

$$238 + 2 = 240$$

$$\begin{array}{r} 6\overline{)\,240\,}(40 \\ \underline{240} \\ \times \end{array}$$

Thus $(238 + 2)$ becomes perfectly divisible by the particular divisor.

For negative integers

$$e.g., -37 \div 5 \quad \frac{-37 - 3 + 3}{5} = \frac{-40 + 3}{5} = \frac{-40}{5} + \frac{3}{5}$$

Here, 8 is quotient and 3 is remainder.

NOTE The remainder is always a non-negative integer.

Rule 1. If a number N is divisible by D then the product of N with any other integral number K is also divisible by D.

$$i.e., \quad \frac{N}{D} \to \text{Remainder } 0$$

$$\Rightarrow \frac{NK}{D} \to \text{Remainder } 0$$

Rule 2. If N_1 and N_2 two different numbers are divisible by D individually, then their sum (*i.e.*, $N_1 + N_2$) and their difference ($N_1 \sim N_2$) is also divisible by D.

Rule 3. If a number N_1 is divisible by N_2 and N_2 is divisible by N_3 then N_1 must be divisible by N_3.

Rule 4. If two numbers N_1 and N_2 are such that they divide mutually each other it means they are same *i.e.*, $N_1 = N_2$ only.

How to Check the Divisibility

We have certain rules to check the divisibility by certain integral numbers. With the help of the following rules it has become easier to know whether a certain number is divisible or not by a particular number without actually dividing the number.

Divisibility by 2

Any integral number whose last digit (*i.e.*, unit digit) is even or in other words the unit digit is divisible by 2. It means any number whose last digit is either 0, 2, 4, 6 or 8, then this number must be divisible by 2.

e.g., 395672, 132, 790, 377754, etc.

Divisibility by 4

If the number formed by last two digits of the given number is divisible by 4, then the actual number must be divisible by 4. *i.e.*, the last two digits of a number can be 00, 04, 08, 12, 16, 20, 24, 28, 32, …, 96.

e.g., 33932, 7958956, 2300, 1996, 3819764280 etc.

Divisibility by 8

If the number formed by last three digits of the given number is divisible by 8, then the actual number must be divisible by 8 *i.e.*, the last three digits of the divisible number can be 000, 008, 016, 024, 032, 040, …, 096, 104, …, 992.

e.g., 8537000, 9317640, 3945080, 23456008, 12345728, 3152408 etc.

Divisibility by 16

If the number formed by last four digits of the given number is divisible by 16, then the actual number must be divisible by 16 *i.e.*, the last 4 digits of the divisible number can be 0000, 0016, 0032, 0048, 0064, 0080, 0096, 0112, 0128, 0144, 0160, …, 0960, 0976, …, 0992, …, 1600, …, 9984.

Divisibility by 32, 64, 128, … can be checked just by checking the last 5, 6, 7, … digit number formed from the given number as in the above cases.

Divisibility by 5

A number is divisible by 5 if and only if the last (*i.e.*, unit) digit is either 0 or 5.

e.g., 5, 10, 15, 20, 25, 30, 35, …, 275, 365, …, 995, 70000, 438915 etc.

Divisibility by 3

If the sum of the digits of the given number is divisible by 3 then the actual number will also be divisible by 3.

e.g., 12375 is divisible by 3 since the sum of digits $1 + 2 + 3 + 7 + 5 = 18$ is divisible by 3.

Similarly 63089154 is also divisible by 3 since the sum of its digits $6 + 3 + 0 + 8 + 9 + 1 + 5 + 4 = 36$ is divisible by 3.

Divisibility by 9

If the sum of the digits of the given number is divisible by 9 then the actual number will also be divisible by 9.

e.g., 7329753 is divisible by 9, since

$$7 + 3 + 2 + 9 + 7 + 5 + 3 = 36 \text{ is divisible by 9.}$$

Similarly the divisibility by 27, 81, 243, … can be checked.

Divisibility by 99

Consider any number. Starting from the right side split the number into the pairs like two digit numbers; if any digit is left unpaired in the left side of the original number take it as a single digit number.

If the sum of all these numbers is divisible by 99, then the given number will also be divisible by 99.

Exp. 1) Test whether 757845 is divisible by 99 or not.

Solution Split the given number starting from the right side as 75 78 45.

Now add all the numbers as 75 + 78 + 45 =198.

Since the sum of all these numbers (198) is divisible by 99, so the given number

757845 is also divisible by 99.

Exp. 2) Test whether 382546692 is divisible by 99 or not.

Solution Split the given number starting from the right side as 3 82 54 66 92

Now add all the numbers as 3+82+54+66+92=297.

Since the sum of all these numbers (297) is divisible by 99, so the given number

362546692 is also divisible by 99.

Exp. 3) Test whether 967845 is divisible by 99 or not.

Solution Split the given number starting from the right side as 96 78 45.

Now add all the numbers as 96 + 78 + 45 =219.

Since the sum of all these numbers (219) is NOT divisible by 99, so the given number 967845 is NOT divisible by 99.

NOTE As you can see that when 219 is divided by 99, it leaves the remainder 21, so the same remainder 21 will be obtained when 967845 is divided by 99.

Divisibility by 999

Consider any number. Starting from the right side split the number into the triplets like three digit numbers; if any digits are left ungrouped in the left side of the original number take the remaining digit(s) as a single/double digit number. If the sum of all these numbers is divisible by 999, then the given number will also be divisible by 999.

Exp. 1) Test whether 579876543 is divisible by 999 or not.

Solution Split the given number starting from the right side as <u>579</u> <u>876</u> <u>543</u>.

Now add all the numbers as 579 + 876 + 543=1998.

Since the sum of all these numbers (1998) is divisible by 999, so the given number 579876543 is also divisible by 999.

Exp. 2) Test whether 34590900485988 is divisible by 999 or not.

Solution Split the given number starting from the right side as <u>34</u> <u>590</u> <u>900</u> <u>485988</u>.

Now add all the numbers as 34 + 590 + 900 + 485 + 988=2997.

Since the sum of all these numbers (2997) is divisible by 999, so the given number 34590900485988 is also divisible by 999.

Exp. 3) Test whether 6800900292 is divisible by 999 or not.

Solution Split the given number starting from the right side as <u>6</u> <u>800</u> <u>900</u> <u>292</u>,

Now add all the numbers as 6 + 800 + 900 + 292=1998.

Since the sum of all these numbers (1998) is divisible by 999, so the given number

6800900292 is also divisible by 999.

Divisibility by 9999

Consider any number. Starting from the right side split the number into the 4-tuples like four digit numbers; if any digits are left untuplled in the left side of the original number take the remaining digit(s) as a single/double/triple digit number. If the sum of all these numbers is divisible by 9999, then the given number will also be divisible by 9999.

Exp. 1) Test whether 652587344739 is divisible by 9999 or not.

Solution Split the given number starting from the right side as <u>6525</u> <u>8734</u> <u>4739</u>.

Now add all the numbers as 6525+ 8734 + 4739=19998.

Since the sum of all these numbers (19998) is divisible by 9999, so the given number 652587344739 is also divisible by 9999.

Exp. 2) Test whether 239976 is divisible by 9999 or not.

Solution Split the given number starting from the right side as <u>23</u> <u>9976</u>.

Now add all the numbers as 23 + 9976=9999.

Since the sum of all these numbers (9999) is divisible by 9999, so the given number 239976 is also divisible by 9999.

Divisibility by Some Special Numbers *e.g.*, 7, 11, 13, 17, 19 etc.

Why are these numbers treated differently for checking the divisibility? Since, if we multiply these numbers by any other number (except 10 or multiples of 10) these numbers can never be divisible by 10. So to make the process easier we bring the given number (*i.e.*, divisor) closer to the multiples of 10 with the difference of 1 (*e.g.*, 9 or 11, 19 or 21, 29 or 31, 39 or 41 etc).

Now if the number is one less than the multiples of 10 (*i.e.*, 9, 19, 29, 39, …) we need to increase it by 1 to make it further multiple of 10. Hence we call it ''one more'' osculator and the value of multiplier of 10 is called the value of ''one more'' osculator.

Similarly if the number is one more than the multiple of 10 (*i.e.*, 11, 21, 31, 41, …, etc) we need to decrease it by 1 to make it the multiple of 10. So we call it 'negative' osculator and the value of multiplier of 10 is the value of 'negative' osculator.

e.g., The given number is 7 (divisor number) then we bring it closer to the multiple of 10 with the difference of 1, then

$$7 \times 3 = 21 = 20 + 1$$

$$\Rightarrow \quad 20 = 21 - 1 = (\underline{2} \times 10) \qquad \ldots\text{(i)}$$

Now in the above expression 2 is the multiplier of 10 and 2 is known as the negative osculator for 7.

Now let us consider the next example of 13 and then bring it closer to the multiples of 10 with a difference of 1.

Thus, $13 \times 3 = 39$ and $39 + 1 = 40 = (4 \times 10) \qquad \ldots\text{(ii)}$

In the above expression 4 is the multiplier of 10 and 4 is the value of 'one more' osculator.

Similarly for 17 we can find the osculator as

$$17 \times 3 = 51 = 50 + 1$$

So, $$50 = 51 - 1 = (\underline{5} \times 10)$$

Here, $\underline{5}$ is the negative osculator for 17.

Similarly for 31 the osculator is negative and the value of negative osculator is 3.

$\because \quad 31 = 30 + 1 \Rightarrow 30 = 31 - 1 \quad \Rightarrow \quad \underline{(3 \times 10)}$

Now we will apply the osculator techniques to check the divisibility by 7, 11, 13, 17, 19, … etc.

Divisibility by 7

To check the divisibility of a number by 7 we apply the following method. Let the number be 133.

Step 1. $133 \Rightarrow \underline{13} - \underline{3} \times 2 = 13 - 6 = 7$

Since 7 is divisible by 7, so the given number 133 will also be divisible by 7. In the above process 2 is multiplied with the last digit is the negative osculator for 7, which I have earlier discussed.

Exp. 1) Check whether 1071 is divisible by 7.

Solution **Step 1.** $\underline{107}\underline{1} \Rightarrow 107 - 1 \times 2 = 105$

Step 2. $\underline{10}\underline{5} \Rightarrow \underline{10} - 5 \times 2 = 0$

Since 0 is divisible by 7 hence the given number 1071 is also divisible by 7.

Exp. 2) Check whether 939715 is divisible by 7.

Solution **Step 1.** $\underline{93971}\underline{5} \Rightarrow 93971 - 5 \times 2 = 93961$

Step 2. $\underline{9396}\underline{1} \Rightarrow 9396 - 1 \times 2 = 9394$

Step 3. $\underline{939}\underline{4} \Rightarrow 939 - 4 \times 2 = 931$

Step 4. $\underline{93}\underline{1} \Rightarrow 93 - 1 \times 2 = 91$

Step 5. $\underline{9}\underline{1} \Rightarrow 9 - 1 \times 2 = 7$

Hence, it is divisible by 7.

NOTE

1. In all the above examples we have to multiply the last digit by the appropriate osculator and then this value will be subtracted from the number formed by the rest digits of the number and this process is continued till you know that the resultant value is divisible by 7. Even you can stop the process in midway when you guess that the obtained value is divisible by 7. For example in the latest problem (example-2) we can stop at step 4 if we know that the 91 is divisible by 7 also we can stop even at step 3 if we have any idea that 931 is divisible by 7.
2. Remember that if the operating osculator is "one more" osculator then we add the product of last digit and one more osculator in the number formed by the rest digit else. We subtract if the operating osculator is 'negative' osculator.

Divisibility by 11

A number is divisible by 11 if the difference between the sum of the digits at odd places and sum of the digits at even places is equal to zero or multiple of 11 (*i.e.,* 11, 22, 33 etc.)

For example : 57945822

Here sum of the digits at odd places $= 2 + 8 + 4 + 7 = 21$ and sum of the digits at even places $= 2 + 5 + 9 + 5 = 21$ and thus the difference $= 0\ (= 21 - 21)$

Hence 57945822 is divisible by 11.

Alternative Approach Consider any number. Starting from the right side split the number into the pairs like two digit numbers; if any digit is left unpaired in the left side of the original number take it as a single digit number.

If the sum of all these numbers is divisible by 11, then the given number will also be divisible by 11.

Exp. 1) Test whether 702845 is divisible by 11 or not.

Solution Split the given number starting from the right side as <u>70</u> <u>28</u> <u>45</u>.

Now add all the numbers as 70 + 28 + 45 =143.

Since the sum of all these numbers (143) is divisible by 11, so the given number <u>70</u> <u>28</u> <u>45</u> is also divisible by 11.

Exp. 2) Test whether 382546692 is divisible by 11 or not

Solution Split the given number starting from thr right side as <u>38</u> <u>25</u> <u>46</u> <u>692</u>

Now add all the numbers as 3 + 82 + 54 + 66 + 92 = 297.

Since the sum of all these numbers (297) is divisible by 11, so the given number 382546692 is also divisible by 11.

Exp. 3) Test whether 967845 is divisible by 11 or not.

Solution Split the given number starting from the right side as <u>96</u> <u>78</u> <u>45</u>.

Now add all the numbers as 96 + 78 + 45 =219.

Since the sum of all these numbers (219) is NOT divisible by 11, so the given number 967845 is NOT divisible by 11.

Exp. 4) Test whether 825466 is divisible by 11 or not, if not, what's remainder?

Solution Split the given number starting from the right side as <u>82</u> <u>54</u> <u>66</u>.

Now add all the numbers as 82+54+66=202.

Since the sum of all these numbers (202) is NOT divisible by 11, so the given number 825466 is also NOT divisible by 11.

Further, since when 202 is divided by 11 it laves remainder 4, so when the original number 825466 is divided by 11 it will leave the same remainder 4.

Divisibility by 111

Consider any number. Starting from the right side split the number into the triplets like three digit numbers; if any digits are left unpaired in the left side of the original number take the remaining digit(s) as a single/double digit number.

If the sum of all these numbers is divisible by 111, then the given number will also be divisible by 111.

***Exp.* 1)** Test whether 579876543 is divisible by 111 or not.

Solution Split the given number starting from the right side as 579 876 543.

Now add all the numbers as 579 + 876 + 543=1998.

Since the sum of all these numbers (1998) is divisible by 111, so the given number 579876543 is also divisible by 111.

***Exp.* 2)** Test whether 34590900485988 is divisible by 111 or not.

Solution Split the given number starting from the right side as 34 590 900 485 988.

Now add all the numbers as 34 + 590 + 900 + 485 + 988=2997.

Since the sum of all these numbers (2997) is divisible by 111, so the given number 34590900485988 is also divisible by 111. To check that whether 2997 is divisible by 111, you can use the same technique as 2997 can be broken up as 2 997, then 2 + 997 = 999. Since 999 is divisible by 111, so 2997 is also divisible by 111. As 2997 is divisible by 111, so 34590900485 is also divisible by 111.

***Exp.* 3)** Test whether 6800900181 is divisible by 111 or not.

Solution Split the given number starting from the right side as 6 800 900 181.

Now add all the numbers as 6 + 800 + 900 + 181=1887.

Since the sum of all these numbers (1887} is divisible by 111, so the given number 6800900181 is also divisible by 111.

Divisibility by 1111

Consider any number. Starting from the right side split the number into the 4-tuples like four digit numbers; if any digits are left unpaired in the left side of the original number take the remaining digit(s) as a single/ double/triple digit number. If the sum of all these numbers is divisible by 1111, then the given number will also be divisible by 1111.

***Exp.* 1)** Test whether 652587344739 is divisible by 1111 or not

Solution Split the given number starting from the right side as 6525 8734 4739.

Now add all the numbers as 6525 + 8734 + 4739=19998.

Since the sum of all these numbers (19998) is divisible by 1111, so the qiven number 652587344739 is also divisible by 1111.

NOTE To check that whether 19998 is divisible by 1111, you can use the same technique as 19998 can be broken up as 1 9998, then 1 + 9998 = 9999. Since, 9999 is divisible by 1111, so 19998 is also divisible by 1111. As 19998 is divisible by 1111, so 652587344739 is also divisible by 1111.

***Exp.* 2)** Test whether 217756 is divisible by 1111 or not.

Solution Split the given number starting from the right side as 21 7756.

Now add all the numbers as 21 + 7756 = 7777.

Since the sum of all these numbers (7777) is divisible by 1111, so the given number 217756 is also divisible by 1111.

Divisibility by 13

Exp. 1) Check whether 2366 is divisible by 13.

Solution **Step 1.** 2366 $\Rightarrow$ $236 + 6 \times 4 = 260$

[Since, the osculator for 13 is 4 and it is 'one more' osculator. So we use addition]

Step 2. 260 $\Rightarrow$ $26 + 0 \times 4 = 26$

Since 26 (or 260) is divisible by 13 hence 2366 is also divisible by 13.

Exp. 2) Check whether 377910 is divisible by 13.

Solution **Step 1.** 377910 $\Rightarrow$ $37791 + 0 \times 4 = 37791$

Step 2. 37791 $\Rightarrow$ $3779 + 1 \times 4 = 3783$

Step 3. 3783 $\Rightarrow$ $378 + 3 \times 4 = 390$

Step 4. 390 $\Rightarrow$ $39 + 0 \times 4 = 39$

Since 39 is divisible by 39. So 377910 is also divisible by 13.

Divisibility by 17

Exp. 1) Find out whether 323 is divisible by 17.

Solution **Step 1.** 32 3 $\Rightarrow$ $32 - 3 \times 5 = 17$

[$\because$ 5 is the negative osculator of 17]

Therefore 323 is divisible by 17.

Exp. 2) Checkout that 12716 is divisible by 17 or not.

Solution **Step 1.** 1271 6 $\Rightarrow$ $1271 - 6 \times 5 = 1241$

1241 1 $\Rightarrow$ $124 - 1 \times 5 = 119$

119 $\Rightarrow$ $11 - 9 \times 5 = -34$

so we can conclude that 12716 is divisible by 17. Since − 34 and 119 both are simply visible that these two numbers are divisible by 17. As I have already mentioned that you can stop your checking process as soon as you can get a number which is easy to know that the particular number is divisible by the given divisor or not. Further you should know that every resultant value in the right hand side might be divisible by the divisor whether you readily recognise it or not.

Divisibility by 19

Exp. 1) Find out whether 21793 is divisible by 19.

Solution **Step 1.** 2179 3 $\Rightarrow$ $2179 + 3 \times 2 = 2185$

[$\because$ 2 is the ''one more'' osculator of 19]

Step 2. 218 5 $\Rightarrow$ $218 + 5 \times 2 = 228$

Step 3. 22 8 $\Rightarrow$ $22 + 8 \times 2 = 38$

Hence 21793 is divisibe by 19.

Shortcut rule for the divisibility by 7, 11 and 13

A number can be divisible by 7, 11 or 13 if and only if the difference of the number formed by the last three digits and the number formed by the rest digits is divisible by 7, 11 or 13 respectively.

For example we have to check that 139125 is divisible by 7 or not.

So we take the difference as given below

$$139 - 125 = 14$$

Since, the difference is divisible by 7. Hence the given number is also divisible by 7.

Exp. 1) Check whether 12478375 is divisible by 13 or not.

Solution **Step 1.** $12478 - 375 = 12103$

Step 2. $12 - 103 = -91$

Since 91 is divisible by 13 hence 12478375 is also divisible by 13.

Divisibility by Composite Numbers e.g., 4, 6, 8, 10, 12, 14, 15 etc.

Divisibility by 6

A number is divisible by 6 only when it is divisible by 2 and 3 both.

So first of all we see that the number is even or not then we check for the divisibility by 3.

Divisibility by 10

A number is divisible by 10 if and only if when it is divisible by both 2 and 5. So it can be easily observed that a number is divisible by 10 must end up with zero(s) at the right end (*i.e.,* last digits) of the given number itself.

NOTE Thus we can say that if there are '*n*' zeros at the end of the given number the number can be divided by 10^n means 10000... *n* zeros.

Divisibility by 12

A number is divisible by 12 only when it is divisible by 4 and 3 both at the same time. So first of all check the divisibility by 4 then by 3

Divisibility by 15

A number is divisible by 15 only when it is divisible by 3 and 5 both simultaneously. So first of all check the number by 5 then by 3. Thus we can conclude that any number which is divisible by a composite number, as mentioned above, must be divisible by all its factors whose L.C.M. is the given divisor.

Introductory Exercise 1.1

1. The number 12375 is divisible by :
(a) 3, 11 and 9 (b) 3 and 11 only
(c) 11 and 9 only (d) 3 and 9 only

2. The least number which must be subtracted from 6708 to make it exactly divisible by 9 is :
(a) 1 (b) 2
(c) 3 (d) 4

3. The smallest number which must be added to 803642 in order to obtain a multiple of 9 is :
(a) 1 (b) 2 (c) 3 (d) 4

4. The divisor when the quotient, dividend and the remainder are respectively 547, 171282 and 71 is equal to :
(a) 333 (b) 323 (c) 313 (d) 303

5. In a problem involving division, the divisor is eight times the quotient and four times the remainder. If the remainder is 12, then the dividend is :
(a) 300 (b) 288
(c) 512 (d) 524

6. 111111111111 is divisible by :
(a) 3 and 37 only
(b) 3, 37 and 11 only
(c) 3, 11, 37 and 111 only
(d) 3, 11, 37, 111 and 1001

7. An integer is divisible by 16 if and only if its last *X* digits are divisible by 16. The value of *X* would be :
(a) three (b) four (c) five (d) six

8. If the number 243x51 is divisible by 9, the value of the digit marked as x would be:
(a) 3 (b) 1 (c) 2 (d) 4

9. Which of the following number is divisible by 999?
(a) 9999999 (b) 99999
(c) 987654321 (d) 145854

10. Which of the following can divide 99999999 exactly?
(a) 9 (b) 9999
(c) 99 (d) Each of (a), (b), and (c)

11. If 24AB4 is divisible by 99, then $A \times B$ is:
(a) 25 (b) 30 (c) 20 (d) 15

12. What is the largest possible two digit number by which 2179782 can be divided ?
(a) 88 (b) 50
(c) 66 (d) 99

13. At least which number must be subtracted from 9999999 so that it will become the multiple of 125?
(a) 124 (b) 4
(c) 24 (d) none of these

14. A number of the form $10^n - 1$ is always divisible by 11 for every *n* is a natural number, when :
(a) *n* is odd (b) *n* is prime
(c) *n* is even (d) can't say

15. Out of the following numbers which is divisible by 132?
(a) 31218 (b) 78520
(c) 38148 (d) 52020

16. If $653xy$ is divisible by 80 then the value of $x + y$ is :
(a) 2 (b) 3
(c) 4 (d) 6

17. The value of k if $k35624$ is divisible by 11 :
(a) 2 (b) 5 (c) 7 (d) 6

18. If $42573k$ is divisible by 72 then the value of k is :
(a) 4 (b) 5
(c) 6 (d) 7

19. How many numbers between 1 and 1000 are divisible by 7?
(a) 777 (b) 142
(c) 143 (d) none of these

20. How many numbers between 55 and 555 including both the extreme values are divisible by 5?
(a) 100 (b) 111
(c) 101 (d) none of these

21. How many numbers are there from 100 to 200 ?
(a) 100 (b) 101
(c) 99 (d) none of these

22. How many numbers are divisible by 3 in the set of numbers 300, 301, 302, ..., 499, 500?
(a) 200 (b) 66
(c) 67 (d) none of these

23. How many numbers are there between 200 and 800 which are divisible by both 5 and 7?
(a) 35 (b) 16
(c) 17 (d) can't be determined

24. In the above question total numbers in the set of numbers $S = \{200, 201, \ldots, 800\}$ which are either divisible by 5 or by 7 is :
(a) 210 (b) 190
(c) 199 (d) can't be determined

25. How many numbers are there in the set $S = \{200, 201, 202, \ldots, 800\}$ which are divisible by neither of 5 or 7?
(a) 411 (b) 412
(c) 410 (d) none of these

26. Total number of numbers lying in the range of 1331 and 3113 which are neither divisible by 2, 3 or 5 is :
(a) 477 (b) 594
(c) 653 (d) none of these

27. Atleast what number must be subtracted from 434079 so that it becomes divisible by 137 ?
(a) 173 (b) 63
(c) 97 (d) can't be determined

28. In the above question, at least what number be added to 434079, so that it will become divisible by (or multiple of) 137 ?
(a) 97 (b) 74
(c) 75 (d) none of these

29. Which one number is closest to 193 which is divisible by 18 is :
(a) 180 (b) 195
(c) 198 (d) 108

30. The product of two numbers $ab7$ and $cd5$ could be, where $ab7$ and $cd5$ are individually three digit numbers :
(a) 8135 (b) 79236
(c) 8735255 (d) none of these

31. When a 3 digit number 984 is added to another 3 digit number $4p3$, we get a four digit number $13q7$, which is divisible by 11. The value of $p + q$ is :
(a) 10 (b) 11
(c) 12 (d) 13

32. When a number divided by 9235, we get the quotient 888 and the remainder 222, such a least possible number is :
(a) 820090 (b) 8200920
(c) 8200680 (d) none of these

33. The number which when divided by 33 is perfectly divisible and closer to 1000 is :
(a) 990
(b) 999
(c) 1023
(d) can't be determined

34. A number which when divided by 32 leaves a remainder of 29. If this number is divided by 8 the remainder will be :
(a) 0 (b) 1
(c) 5 (d) 3

35. A number when divided by 5 leaves a remainder of 4, when the double (*i.e.,* twice) of that number is divided by 5 the remainder will be :
(a) 0
(b) 1
(c) 3
(d) can't be determined

36. When a number 'N' is divided by a proper divisor 'D' then it leaves a remainder of 14 and if the thrice of that number *i.e.,* $3N$ is divided by the same divisor D, the remainder comes out to be 8. Again if the 4 times of the same number *i.e.,* '$4N$' is divided by D the remainder will be :
(a) 35 (b) 22
(c) 5 (d) can't be determined

37. A number when divided by 5 gives a number which is 8 more than the remainder obtained on dividing the same number by 34. Such a least possible number is:
(a) 175 (b) 75
(c) 680 (d) does not exist

38. When a natural number divided by a certain divisor, we get 15 as a remainder. But when the 10 times of the same number is divided by the same divisor we get 6 as a remainder. The maximum possible number of such divisors is :

(a) 6 (b) 7
(c) 15 (d) can't be determined

39. A certain number 'C' when divided by N_1 it leaves a remainder of 13 and when it is divided by N_2 it leaves a remainder of 1, where N_1 and N_2 are the positive integers. Then the value of $N_1 + N_2$ is, if $\frac{N_1}{N_2} = \frac{5}{4}$:

(a) 36
(b) 27
(c) 54
(d) can't be determined uniquely

40. In the above problem the value of c is :

(a) $50 < c < 100$
(b) any multiple of 11
(c) $20 < c < 50$
(d) can't be determined

41. In how many parts a rod of length 19.5 m can be broken of equal length of 65 cm?

(a) 20 (b) 30
(c) 3 (d) 130

42. A six digit number which is consisting of only one digits either 1, 2, 3, 4, 5, 6, 7, 8 or 9, e.g., 111111, 222222... etc. This number is always divisible by :

(a) 7 (b) 11
(c) 13 (d) all of these

43. The maximum possible difference between the 4 digit numbers formed by using the 4 different digits 1, 2, 3, 5 is :

(a) 4086 (b) 5076
(c) 4386 (d) 3242

44. The sum of all digits except the unity that can be substituted at the place of k inorder to be divisible by 8 in the number 23487k2 :

(a) 5 (b) 14
(c) 9 (d) none of these

45. A certain number N when multiplied by 13, the resultant values consists entirely of sevens. The value of N is :

(a) 123459 (b) 58829
(c) 59829 (d) none of these

46. A teacher who teaches online at Lamamia wrote a 90 digit positive number 112222333333... in his laptop. Then he inserted a three-digit number between any two distinct digits. Now the new number cannot be perfectly divisible by 11. Find the number of possible values of three digit numbers.

(a) 819 (b) 781
(c) 881 (d) none of these

47. A natural number is divisible by 1125, which consists of only 0s and 1s. Minimum how many 0s and 1s are there in such a number?

(a) 5,9 (b) 7,5 (c) 3,3 (d) 3,9

48. A ten-digit number containing each distinct digit only once. The ten-digit number is divisible by 10. If the last digit is eliminated, the remaining number is divisible by 9. If the last 2 digits are eliminated, the remaining number is divisible by 8. If the last three digits are eliminated, the remaining number is divisible by 7. And so on. Find the number.

(a) 1234567890 (b) 2436517890
(c) 1832547690 (d) 3816547290

49. What is the smallest 9 digit number containing all the non-zero digits 1, 2, ..., 9, which is divisible by 99?

(a) 125364789 (b) 124365879
(c) 123475689 (d) none of these

50. An uninitiated student once visited a test prep portal www. Lamamia.in to assess her quantitative ability for SSC CGL. She took a test in which there were total 18 problems each with four options—A, B, C and D. After glancing through the whole test paper she realized that she was not able to crack even a single problem, so she marked the choice A in all the problems. Then after a while she changed the answers by marking B in each third problem, starting with the third problem. Unsatisfied with her answers, she again changed the answers by marking C in each second problem, starting with the second problem. In her final attempt to guess the answers, she changed the answers by marking D in each ninth problem, starting with the ninth problem. While. changing the answers she moves from the first to the last problem in an orderly way. What are the numbers of final answers that she marked in terms of A, B, C and D, respectively?

(a) 6, 2, 1,9 (b) 6, 2, 8, 2
(c) 2, 4, 6, 6 (d) 4, 6, 2, 6

1.2 Square and Cubes

SQUARE TABLE

1	1	21	441	41	1681	61	3721	81	6561
2	4	22	484	42	1764	62	3844	82	6724
3	9	23	529	43	1849	63	3969	83	6889
4	16	24	576	44	1936	64	4096	84	7056
5	25	25	625	45	2025	65	4225	85	7225
6	36	26	676	46	2116	66	4356	86	7396
7	49	27	729	47	2209	67	4489	87	7569
8	64	28	784	48	2304	68	4624	88	7744
9	81	29	841	49	2401	69	4761	89	7921
10	100	30	900	50	2500	70	4900	90	8100
11	121	31	961	51	2601	71	5041	91	8281
12	144	32	1024	52	2704	72	5184	92	8464
13	169	33	1089	53	2809	73	5329	93	8649
14	196	34	1156	54	2916	74	5476	94	8836
15	225	35	1225	55	3025	75	5625	95	9025
16	256	36	1296	56	3136	76	5776	96	9216
17	289	37	1369	57	3249	77	5929	97	9409
18	324	38	1444	58	3364	78	6084	98	9604
19	361	39	1521	59	3481	79	6241	99	9801
20	400	40	1600	60	3600	80	6400	100	10,000

Squaring Techniques

Dear students let me tell you a simple secret of Number System. It is nothing but knowing the properties of numbers and so knowing the squares of numbers makes you smarter than the ones who don't know the squares or the ones who can't calculate the squares orally in a few seconds.

A couple of techniques are illustrated in this book that will definitely help you calculate the squares faster than many of your competitors. My only request is that do not undermine the role of squares due to your ignorance or impatience.

1. **Square of the numbers whose unit (*i.e.,* last) digit is zero (0) :** Any number whose last digit is zero, we double the number of zeros at the right side of the number and in the left of the zeros we write the square of the non-zero numbers (or rest digits) which is in the left of the zero(s)

   e.g., square of 40 :

   $(40)^2 \Rightarrow$ **Step 1.** 00

   Step 2. $4^2 = 16$

   Step 3. 1600

   Similarly, square of 700 :

   $(700)^2 \Rightarrow$ **Step 1.** 0000

   Step 2. $(7)^2 = 49$

   Step 3. 490000

   Square of 13000 :

   $(13000)^2 \Rightarrow$ **Step 1.** 000000

   Step 2. $(13)^2 = 169$

   Step 3. 169000000

   Square of 21000 :

   $(21000)^2 \Rightarrow$ **Step 1.** 000000

   Step 2. $(21)^2 = 441$

   Step 3. 441000000

2. **Square of the numbers whose unit digit is 5 :**

   To calculate the square of the number whose unit digit is 5, first of all, we write 25 as the last two digits of the square of the given number. For remaining digits which has to be written to the left of 25 we write the product of the number formed by remaining digits of the given number and the its successive number.

For example :

(i) The square of 35 :

$(35)^2 \Rightarrow$ **Step 1.** 25

Step 2. $3 \times (3+1) = 12$

Step 3. 1225

(ii) The square of 65 :

$(65)^2 \Rightarrow$ **Step 1.** 25

Step 2. $6 \times (6+1) = 42$

Step 3. 4225

(iii) The square of 75 :

$(75)^2 \Rightarrow$ **Step 1.** 25

Step 2. $7 \times (7+1) = 56$

Step 3. 5625

(iv) The square of 125 :

$(125)^2 \Rightarrow$ **Step 1.** 25

Step 2. $12 \times (12+1) = 156$

Step 3. 15625

(v) The square of 1405

Step 1. 25

Step 2. $140 \times (140+1) = 19740$

Step 3. 1974025

(vi) The square of 139005

$(139005)^2 \Rightarrow$ **Step 1.** 25

Step 2. $13900 \times (13900+1) = 193223900$

Step 3. 19322390025

(vii) The square of 155 :

$(155)^2 \Rightarrow$ **Step 1.** 25

Step 2. $15 \times (15+1) = 240$

Step 3. 24025

3. (i) **Square of the numbers ending with the digit 1 :** These numbers are very easy to solve. To get the square of any such number just write the square of the previous number and then add the previous number and the number whose square is being asked.

For example the square of 21

$$(21)^2 = (20)^2 + (20+21) = 441$$

Similarly, $(31)^2 = (30)^2 + (30+31) = 961$

and $(41)^2 = (40)^2 + (40+41) = 1681$

and $(81)^2 = (80)^2 + (80+81) = 6561$

and $(131)^2 = (130)^2 + (130+131)$

$= 16900 + 261 = 17161$

and $(151)^2 = (150)^2 + (150+151)$

$= 22500 + 301 = 22801$

and $(1111)^2 = (1110)^2 + (1110+1111)$

$= (1110)^2 + 2221$

$= 1232100 + 2221 = 1234321$

(ii) **Square of the numbers whose unit digit is 9 :** It is very similar to the previous case. The difference is that here we have to subtract instead of addition. For example the square of 39

$$(39)^2 = (40)^2 - (39+40)$$
$$= 1600 - 79 = 1521$$

Similarly, $(49)^2 = (50)^2 - (49+50)$

$= 2500 - 99 = 2401$

and $(89)^2 = (90)^2 - (89+90)$

$= 8100 - 179 = 7921$

and $(139)^2 = (140)^2 - (139+140)$

$= 19600 - 279 = 19321$

and $(249)^2 = (250)^2 - (249+250)$

$= 62500 - 499 = 62001$

NOTE In the case of unit digit of 9, we subtract from the base square because base is higher than the number whose square is to be calculated.

(iii) **Square of the numbers whose last digit is 2 or 8 :**

For example $(52)^2 = (50)^2 + 2(50+52)$

$= 2500 + 204 = 2704$

Similarly, $(112)^2 = (110)^2 + 2(110+112)$

$= 12100 + 444 = 12544$

and $(38)^2 = (40)^2 - 2(38+40)$

$= 1600 - 156 = 1444$

and $(88)^2 = (90)^2 - 2(88+90)$

$= 8100 - 356 = 7744$

(iv) **Square of the numbers whose unit digit is 3 or 7 :**

For example : $(23)^2 = (20)^2 + 3(20+23)$

$= 400 + 129 = 529$

Similarly, $(53)^2 = (50)^2 + 3(50+53)$

$= 2500 + 309 = 2809$

and $(123)^2 = (120)^2 + 3(120+123)$

$= 14400 + 729 = 15129$

and $(47)^2 = (50)^2 - 3(47+50)$

$= 2500 - 291 = 2209$

and $(137)^2 = (140)^2 - 3(137 + 140)$

$= 19600 - 831 = 18769$

and $(2347)^2 = (2350)^2 - 3(2347 + 2350)$

$= 5522500 - 4691 = 5517809$

(v) **Square of the numbers which ends with 4 or 6 :**

For example : $(34)^2 = (30)^2 + 4(30 + 34)$

$= 900 + 256 = 1156$

or $(34)^2 = (35)^2 - (34 + 35)$

$= 1225 - 69 = 1156$

Similarly, $(36)^2 = (35)^2 + (35 + 36)$

$= 1225 + 71 = 1296$

and $(126)^2 = (125)^2 + (125 + 126)$

$= 15625 + 251 = 15876$

NOTE In general square of any number can be found by using the square of any convenient number as base square *e.g.,* the numbers whose unit digit is either 0 or 5, since the square of these numbers is easy to find and learn. However we can calculate the square by considering any base square. For example, if

$(32)^2 = (30)^2 + 2(30 + 32) = 1024$

or $(32)^2 = (31)^2 + (31 + 32)$

$= 961 + 63 = 1024$

or $(32)^2 = (33)^2 - (32 + 33)$

$= 1089 - 65 = 1024$

or $(32)^2 = (35)^2 - 3(32 + 35)$

$= 1225 - 201 = 1024$

Now, it can be generalized as :

If N is the number whose square is to be calculated and B is the base and d is difference between N and B then $(N)^2 = (B)^2 + d(B + N)$ when $B < N$

or $(N)^2 = (B)^2 - d(B + N)$ when $B > N$

Properties of Squares

(i) If the unit digit of the number is zero, then the unit digit of the square of this number will also be zero and the number of zeros will be double in the square than that of its root. *e.g.,* $(60)^2 = 3600$, $(130)^2 = 16900$

(ii) If the unit digit of the number is 5, then the unit digit of its square is also 5 and the number formed by last two digits is 25.

e.g., $(35)^2 = 1225$, $(45)^2 = 2025$, $(55)^2 = 3025$ etc.

(iii) If the unit digit of any number is 1 or 9, then the unit digit of the square of its number is always 1.

e.g., $(71)^2 = 5041$, $(31)^2 = 961$, $(19)^2 = 361$

(iv) If the unit digit of any number is 2 or 8, then the unit digit of the square of its number is always 4.

(v) If the unit digit of any number is 3 or 7, then the unit digit of its square is always 9.

e.g., $(23)^2 = 529$,

$(27)^2 = 729$

(vi) If the unit digit of any number is 4 or 6, then the unit digit of its square is always 6.

e.g., $(26)^2 = 676$,

$(24)^2 = 576$,

$(14)^2 = 196$,

$(16)^2 = 256$ etc.

(vii) The square of any number is always positive irrespective of the nature of the given number.

(viii) The numbers with unit digit 0, 1, 5 and 6 always give the same unit digits respectively, on squaring.

(ix) 2, 3, 7 and 8 never appear as unit digit in the square of a number.

Perfect Square : The square of any natural number is known as perfect square *e.g.,* 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, ... etc. are the squares of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ... etc.

Square root

A number n is called the square root of a number $n^2 = (n \times n)$ where n^2 is obtained by multiplying the number n with itself only once.

The symbol for square root is '$\sqrt{\ }$'.

For example

the square root of 4 is $\sqrt{4} = 2$,

the square root of 9 is $\sqrt{9} = 3$, and

the square root of 25 is $\sqrt{25} = 5$, etc.

Generally there are two methods for finding the square root :

(i) Prime Factorisation method

(ii) Division method

(i) *Factorisation Method*

In this method first we find out the prime factors and then we pair them as given below.

Exp. 1) Find the square root of 3600.

Solution The factors of 3600

$= \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{3 \times 3} \times \underline{5 \times 5}$

So the square root of 3600 *i.e.,*

$\sqrt{3600} = 2 \times 2 \times 3 \times 5$

$= 60$

2	3600
2	1800
2	900
2	450
5	225
5	45
3	9
3	3
	1

Exp. 2) Find the square root of 144.

Solution Factors of $144 = \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{3 \times 3}$

So $\sqrt{144} = 2 \times 2 \times 3$

$\sqrt{144} = 12$

2	144
2	72
2	36
2	18
3	9
3	3
	1

Exp. 3) Find the square root of 10404.

Solution The factors of 10404

$= \underline{2 \times 2} \times \underline{3 \times 3} \times \underline{17 \times 17}$

$\therefore \quad \sqrt{10404} = 2 \times 3 \times 17$

or $\quad \sqrt{10404} = 102$

2	10404
2	5202
3	2601
3	867
17	289
17	17
	1

(ii) *Division Method*

In this method first of all we make pairs from the right side towards left and then solve as given below.

Exp. 1) Find the square root of 3600.

Solution

	60
6	$\underline{36}\ \underline{00}$
6	36
120	00 00
0	00 00

$\therefore \quad \sqrt{3600} = 60$

Exp. 2) Find the square root of 10404.

Solution

	102
1	1 $\underline{04}$ $\underline{04}$
1	1
202	0 04 04
02	4 04
	× ×

$\therefore \quad \sqrt{10404} = 102$

Exp. 3) Find the square root of 15876.

Solution

	126
1	1 $\underline{58}$ $\underline{76}$
1	1
22	58
2	44
246	1476
6	1476
	×

$\therefore \quad \sqrt{15876} = 126$

Exp. 4) Find the square root of 120409.

Solution

	347
3	$\underline{12}$ $\underline{04}$ $\underline{09}$
3	9
64	304
4	256
687	4809
7	4809
	×

$\therefore \quad \sqrt{120409} = 347$

Exp. 5) Find the square root of 5793649.

Solution

	2407
2	5 $\underline{79}$ $\underline{36}$ $\underline{49}$
2	4
44	179
4	176
4807	33649
07	33649
	× × × ×

$\therefore \quad \sqrt{5793649} = 2407$

NOTE

If $(11)^2 = 121$, then $(101)^2 = 10201$

$(21)^2 = 441$, then

$(201)^2 = 40401$ and so on...

If $(11)^2 = 121$, then $(1001)^2 = 1002001$

If $(31)^2 = 961$, then $(3001)^2 = 9006001$

If $(12)^2 = 144$, then $(1002)^2 = 1004004$ and so on

Go in depth and find the limitation of this logic ...

NOTE

$1^2 = 1$

$11^2 = 121$

$111^2 = 12321$

$1111^2 = 1234321$

$11111^2 = 123454321$

$111111^2 = 12345654321$

… … … … …

Cubing Techniques

If a number is multiplied by itself twice, then the resultant value is called as the cube of that number. For example $n \times n \times n = n^3$, read as n cube.

If we say that what is the cube of 4, then it means

$$4^3 = 4 \times 4 \times 4 = 64$$

Similarly, $\quad 7^3 = 7 \times 7 \times 7 = 343$ and $8^3 = 8 \times 8 \times 8 = 512$

But, for the greater (or larger) number sometimes it becomes tedious to calculate the cube very quickly. So use a new approach to calculate the cube of larger numbers from 11 to 99.

Since you can easily calculate the cubes of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, 40, 50, 60, 70, 80, 90 and 100 etc. So you can leverage it for further calculation.

Method to find cube of two digit numbers

Step 1. Write down the cube of tens digit.

Step 2. Write down the resultant value in the same ratio as the ratio of tens and unit digit in four terms as a G.P.

Step 3. Write down twice the value obtained in the second and third term below the second and third term respectively.

Step 4. Now you can add all the numbers in the prescribed manner as given in the following examples :

NOTE While doing the addition (in step 4) subscripts should be treated as a "carryover" for the next term.

Exp. 1) $(11)^3 \Rightarrow$ Step 1. $1^3 = 1$

Step 2. $\underline{1\ 1\ 1\ 1}$

Step 3. $1\ 1\ 1\ 1$

$\underline{+2\ 2}$

Step 4. $1\ 3\ 3\ 1$

$\therefore \quad (11)^3 = 1331$

Exp. 2) $(12)^3 \Rightarrow$ Step 1. $1^3 = 1$

Step 2. $\underline{1\ 2\ 4\ 8}$

Step 3. $1\ 2\ 4\ 8$

$\underline{4\ 8}$

Step 4. $1\ 7\ 2\ 8$

$\therefore \quad (12)^3 = 1728$

Exp. 3) $(13)^3 \Rightarrow$ Step 1. $1^3 = 1$

Step 2. $\underline{1 \quad 3 \quad 9 \quad 27}$

Step 3. $1 \quad 3 \quad 9 \quad 27$

$\underline{6 \quad 18}$

Step 4. $2 \quad 1 \quad {}_{2}9 \quad {}_{2}7$

$\therefore \quad (13)^3 = 2197$

Exp. 4) $(25)^3 \Rightarrow$ Step 1. $2^3 = 8$

Step 2. $\underline{8 \quad 20 \quad 50 \quad 125}$

Step 3. $8 \quad 20 \quad 50 \quad 125$

$\underline{40 \quad 100}$

Step 4. $15 \quad {}_{7}6 \quad {}_{16}2 \quad {}_{12}5$

$\therefore \quad (25)^3 = 15625$

Exp. 5) $(26)^3 \Rightarrow$ Step 1. $2^3 = 8$

Step 2. $\underline{8 \quad 24 \quad 72 \quad 216}$

Step 3. $8 \quad 24 \quad 72 \quad 216$

$\underline{+ \quad 48 \quad 144}$

Step 4. $17 \quad {}_{9}5 \quad {}_{23}7 \quad {}_{21}6$

$\therefore \quad (26)^3 = 17576$

Exp. 6) $(27)^3 \Rightarrow$ Step 1. $2^3 = 8$

Step 2. $\underline{8 \quad 28 \quad 98 \quad 343}$

Step 3. $8 \quad 28 \quad 98 \quad 343$

$\underline{+ \quad 56 \quad 196}$

Step 4. $19 \quad {}_{11}6 \quad {}_{32}8 \quad {}_{34}3$

$\therefore \quad (27)^3 = 19683$

Exp. 7) $(53)^3 \Rightarrow$ Step 1. $5^3 = 125$

Step 2. $\underline{125 \quad 75 \quad 45 \quad 27}$

Step 3. $125 \quad 75 \quad 45 \quad 27$

$\underline{+ \quad 150 \quad 90}$

Step 4. $148 \quad {}_{23}8 \quad {}_{13}7 \quad {}_{2}7$

$\therefore \quad (53)^3 = 148877$

Exp. 8) $(67)^3 \Rightarrow$ Step 1. $(6)^3 = 216$

Step 2. $\underline{216 \quad 252 \quad 294 \quad 343}$

Step 3. $216 \quad 252 \quad 294 \quad 343$

$\underline{+ \quad 504 \quad 588}$

Step 4. $300 \quad {}_{84}7 \quad {}_{91}6 \quad {}_{34}3$

$\therefore \quad (67)^3 = 300763$

Exp. 9) $(94)^3 \Rightarrow$ Step 1. $(9)^3 = 729$

Step 2. $\underline{729 \quad 324 \quad 144 \quad 64}$

Step 3. $729 \quad 324 \quad 144 \quad 64$

$\underline{+ \quad 648 \quad 288}$

Step 4. $830 \quad {}_{101}5 \quad {}_{43}8 \quad {}_{6}4$

$\therefore \quad (94)^3 = 830584$

Exp. 10) $(88)^3 \Rightarrow$ Step 1. $(8)^3 = 512$

Step 2. $\underline{512 \quad 512 \quad 512 \quad 512}$

Step 3. $512 \quad 512 \quad 512 \quad 512$

$\underline{+ \quad 1024 \quad 1024}$

Step 4. $681 \quad {}_{169}4 \quad {}_{158}7 \quad {}_{51}2$

$\therefore \quad (88)^3 = 681472$

Exp. 11) $(55)^3 \Rightarrow$ Step 1. $(5)^3 = 125$

Step 2. $\underline{125 \quad 125 \quad 125 \quad 125}$

Step 3. $125 \quad 125 \quad 125 \quad 125$

$\underline{+ \quad 250 \quad 250}$

Step 4. $166 \quad {}_{41}3 \quad {}_{38}7 \quad {}_{12}5$

$\therefore \quad (55)^3 = 166375$

Exp. 12) $(72)^3 \Rightarrow$ Step 1. $(7)^3 = 343$

Step 2. $\underline{343 \quad 98 \quad 28 \quad 8}$

Step 3. $343 \quad 98 \quad 28 \quad 8$

$\underline{+ \quad 196 \quad 56}$

Step 4. $373 \quad {}_{30}2 \quad {}_{8}4 \quad 8$

$\therefore \quad (72)^3 = 373248$

Cube Roots

The cube root of a number n is expressed as $\sqrt[3]{n}$ or $(n)^{1/3}$ or in other words $n = (n)^{1/3} \times (n^{1/3}) \times (n^{1/3})$

For example, cube root of 27 *i.e.,* $\sqrt[3]{27} = 3$

and $\quad \sqrt[3]{343} = 7$ and $\sqrt[3]{64} = 4$

Exp. 1) Find the cube root of 125.

Solution $\because \quad 125 = \underline{5 \times 5 \times 5}$

$\therefore \quad \sqrt[3]{125} = 5$

$$\begin{array}{r|l} 5 & 125 \\ \hline 5 & 25 \\ \hline 5 & 5 \\ \hline & 1 \end{array}$$

Exp. 2) Find the cube root of 1728.

Solution $1728 = \underline{2 \times 2 \times 2} \times \underline{2 \times 2 \times 2} \times \underline{3 \times 3 \times 3}$

$\sqrt[3]{1728} = 2 \times 2 \times 3$

$\sqrt[3]{1728} = 12$

$$\begin{array}{r|l} 2 & 1728 \\ \hline 2 & 864 \\ \hline 2 & 432 \\ \hline 2 & 216 \\ \hline 2 & 108 \\ \hline 2 & 54 \\ \hline 3 & 27 \\ \hline 3 & 9 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

NOTE In this method first we factorize the given numbers then we make the triplet of the factors and thus we multiply these triplets to get the cube root of the given number.

Exp. 3) On 14th November, in my school, each child received as many packs of chocolates as there were total number of the students in the school. Further, each pack of chocolates contains as many chocolates as there were the total number of packs which a child had. Total how many chocolates have been distributed among all the children of school :

(a) 729 (b) 196
(c) 961 (d) can't be determined

Solution (*d*) is the appropriate answer since we don't have sufficient data to calculate.

Exp. 4) In the above question if there were 8 packs of chocolates with every child. Then the total number of chocolates distributed among them was :

(a) 512 (b) 64
(c) 256 (d) can't be determined

Solution Let there are n children it means each child has n packs of chocolates and since every pack contains n chocolates. Therefore total number of chocolates

$$= n \times n \times n = n^3$$

Hence total number of chocolates $= (8)^3 = 512$

Thus option (a) is correct.

Introductory Exercise 1.2

1. The square root of 6280036 is :
(a) 1308 (b) 2903
(c) 2506 (d) none of these
2. The square root of 1296 is :
(a) 33 (b) 44 (c) 34 (d) 36
3. The square root of 7744 is :
(a) 94 (b) 88
(c) 77 (d) none of these
4. The square root of 56169 is :
(a) 359 (b) 323
(c) 227 (d) none of these
5. The square root of 1238578 is :
(a) 3254 (b) 3724
(c) 3258 (d) None of these
6. The least number by which we multiply to the 11760, so that we can get a perfect square number :
(a) 2
(b) 3
(c) 5
(d) none of the above
7. In the above question, by which least possible number we divide to the 11760 so the resultant number becomes a perfect square :
(a) 3
(b) 15
(c) 7
(d) can't be determined
8. The least possible positive number which should be added to 575 to make a perfect square number is :
(a) 0 (b) 1
(c) 4 (d) none of these
9. The least possible number which must be subtracted from 575 to make a perfect square number is :
(a) 5 (b) 50
(c) 46 (d) 37
10. The square of a number 'A' is the sum of the square of other two numbers 'B' and 'C'. Where $5B = 12C$ and B, C are positive numbers. The least possible positive value of A is :
(a) 10 (b) 12
(c) 13 (d) 16

11. Lieutenant Kalia when arranged all his 1500 soldiers in such a way that the number of soldiers in a line were the same as there were the number of lines. So he was left with 56 soldiers, who were not a part of this arrangement. The number of lines in this arrangement is :

(a) 44 (b) 36
(c) 38 (d) none of these

12. $\sqrt{289} \div \sqrt{x} = \frac{1}{5}$, then the value of x is :

(a) $\frac{17}{25}$ (b) $\frac{34}{35}$
(c) 235 (d) 7225

13. Out of the following statements which one is incorrect?

(a) $\sqrt{5184} = 72$ (b) $\sqrt{15625} = 125$
(c) $\sqrt{1444} = 38$ (d) $\sqrt{1296} = 34$

14. If $2 * 3 = \sqrt{13}$ and $3 * 4 = 5$, then the value of $5 * 12$ is

(a) 17 (b) $\sqrt{29}$
(c) 21 (d) 13

15. If $a * b * c = \sqrt{\frac{(a+2)(b+3)}{(c+1)}}$, then the value of $(6 * 15 * 3)$ is :

(a) 6 (b) 3
(c) 4 (d) can't be determined

16. The smallest number that must be added to 1780 to make it a perfect square is :

(a) 69 (b) 156
(c) 49 (d) 59

17. If $\frac{100\sqrt{25}}{\sqrt{25} + x} = 50$ then the value of x is :

(a) 25 (b) $\frac{1}{\sqrt{25}}$ (c) $\sqrt{25}$ (d) $\frac{1}{25}$

18. The least possible number which we should add to 1720 to make a perfect cube number is :

(a) 0 (b) 1
(c) 8 (d) 7

19. The least positive number which is subtracted from 1369 to make it a perfect cube is :

(a) 17 (b) 38
(c) 34 (d) none of these

20. The least possible natural number by which if we multiply to the 1372, we get a perfect cube number is :

(a) 2 (b) 3
(c) 5 (d) can't be determined

21. The least possible number by which if we divide 1372, it will become a perfect cube number is :

(a) 2 (b) 7 (c) 3 (d) 4

1.3 **Basic Numbers**

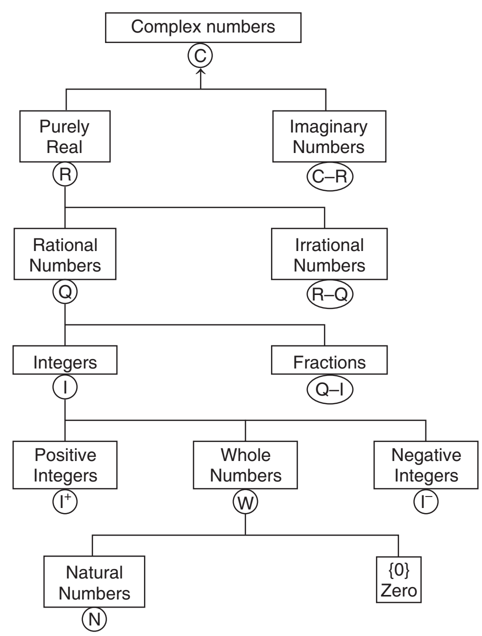
It has been said by a great mathematician that ''Mathematics is the queen of Science and Arithmetic is the queen of Mathematics.'' Arithmetic is the maths of numerals and digits and so the other branches of Maths are dependent upon Arithmetic.

Either the counting of the currency or counting of the stars in all the ways it has been a very popular subject especially among Indians.

One thing which is very well known that the great Indian Mathematician Aryabhatta has introduced '0' (zero) in the counting numbers and the whole world has adopted it. In fact, a computer is dead if this figure does not exist. So one can say zero has revolutionised the whole world.

Here we will discuss all sorts of numbers, their properties, applications besides the different number systems used widely.

The following chart briefly illustrates the different kinds of numbers with their relationship.



Natural Numbers

The counting numbers 1, 2, 3, 4, 5, … are called the natural numbers and are denoted by N, As you may like to recall that a child when he/she is very young he/she starts counting his/her toys, sweets, fruits, etc. as 1, 2, 3, 4, 5, …, as he/she has no idea about 0 (zero) and decimal numbers such as 1.5, 2.75 or 3.33 or any other type of numbers. That's why probably the numbers 1, 2, 3, 4, 5, … are called natural numbers.

Natural numbers are represented by N ,where $N = \{1, 2, 3, \dots\}$.

All natural numbers are the positive integers.

Properties of Natural Numbers

1. **Successor :** The next natural number just after any natural number n is called its successor 'n^+' where $n^+ = n + 1$ for example the successor of 2 is 3, successor of 6 is 7 etc.
2. **Closure law :** For any two natural numbers a and b
$$(a + b) \in N \quad \text{and} \quad (a \times b) \in N$$
e.g., $\mathbf{3 + 4 = 7 \in N}$ **and** $\mathbf{3 \times 4 = 12} \in N$
3. **Commutative law :** For any two natural numbers a and b
$$a + b = b + a \text{ and } a \times b = b \times a$$
e.g., $5 + 6 = 6 + 5 = 11$
and $5 \times 6 = 6 \times 5 = 30$
4. **Associative law :** For any three natural numbers
$$(a + b) + c = a + (b + c)$$
$$(a \times b) \times c = a \times (b \times c)$$
e.g., $(7 + 8) + 9 = 7 + (8 + 9) = 24$
$(7 \times 8) \times 9 = 7 \times (8 \times 9) = 504$
5. **Multiplicative Identity :** 1 is the multiplicative identity of every natural number as
$$4 \times 1 = 4$$
$$5 \times 1 = 5$$
$$10 \times 1 = 10$$
$$17 \times 1 = 17$$
6. **Cancellation law :** For any three natural numbers a, b, c
$$a + b = c + b \Rightarrow a = c$$
and $a \times b = c \times b \Rightarrow a = c$
7. **Distributive law :** For any three natural numbers a, b, c
$$a \times (b + c) = a \times b + a \times c$$
but $a + (b \times c) \neq (a + b) \times (a + c)$
e.g., $3 \times (4 + 5) = 3 \times 4 + 3 \times 5 = 27$
8. **Trichotomy law** If there are any two natural numbers a and b then there exists one and only one relation necessarily is
(i) $a > b$ (ii) $a = b$ (iii) $a < b$

Even Numbers

All the natural numbers which are divisible by 2 are known as even numbers *e.g.,* 2, 4, 6, 8, 10, … .

Odd Numbers

All the natural numbers which are not divisible by 2 are known as 'Odd numbers' *e.g.,* 1, 3, 5, 7, …

Important Note

Even + Even = Even
Even – Even = Even
Even × Even = Even
Even ÷ Even = Even of odd

Odd + Odd = Even
Odd – Odd = Even
Odd × Odd = Even
Odd ÷ Odd = Even

Even + Odd = Odd
Even – Odd = Odd

Even × Odd = Even
Even ÷ Odd = Even

Odd + Even = Odd
Odd – Even = Odd
Odd × Even = Even
Odd ÷ Even = (never divisible)

$(\text{even})^{\text{even/odd}}$ = Even
$(\text{odd})^{\text{odd/even}}$ = Odd

Prime Numbers

Except 1 each natural number which is divisible by only 1 and itself is called as prime number *e.g.,* 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ... etc.

- There are total 25 prime numbers upto 100
- There are total 46 prime numbers upto 200
- 2 is the only even prime number and the least prime number.
- 1 is neither prime nor composite number.
- There are infinite prime numbers.
- A list of all the prime numbers upto 100 is given below.

Table of Prime Numbers $(1 - 100)$:

2	11	23	31	41	53	61	71	83	97
3	13	29	37	43	59	67	73	89	
5	17			47			79		
7	19								

How to test whether a number is prime or not : To test a number n take the square root of n and consider as it is , if it is a natural number otherwise just increase the square root of it to the next natural number. Then divide the given number by all the prime numbers below the square root obtained. If the number is divisible by any of these prime numbers then it is not a prime number else it is a prime number.

Exp.) Check that whether 241 is prime.

Solution When we take the square root of 241 it is approximate 15, so we consider it 16. Now we divide 241 by all the prime numbers below 16 *viz.,* 2, 3, 5, 7, 11, 13
Since 241 is not divisible by any one of the prime numbers below 16. So it is a **prime number.**

NOTE Any digit if it is written continuously 3 times, 6 times, 9 times ... etc. then it is divisible by 3
e.g., 111; 555555; 777777; 222222222; 888, 222 etc.

Co-prime Numbers

Two natural numbers are called co-prime (or relatively prime) numbers if they have no common factor other than 1 or in other words. The highest common factor *i.e.,* H.C.F. between co-prime numbers is 1. *e.g.,* (15, 16), (14, 25), (8, 9), (13, 15) etc.

NOTE It is not necessary that the numbers involved in the pair of co-primes will be prime even they can be composite numbers as seen in the above examples.

Twin Primes

When the difference between any two prime number is 2, these prime numbers are called twin primes.

e.g., (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73), (101, 103), (107, 109), (137, 139), etc.

Composite Numbers

A number other than one which is not a prime number is called a composite number. It means it is divisible by some other number(s) other than 1 and the number itself. *e.g.,* 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26...

Every natural number except 1 is either prime or composite.

Whole Numbers

The extended set of natural numbers in which '0' is also included is called as the set of whole numbers and is denoted by

$$W = \{0, 1, 2, 3, 4, 5, \dots\}$$

NOTE '0' (zero) is an even number.

Consecutive Numbers

A series of numbers in which the next number is 1 more than the previous number or the predecessor number is 1 less than the successor or just they can be differed by 1

e.g., 10, 11, 12, or 17, 18, 19, 20, 21 or 717, 718, 719, 720, 721... etc.

Perfect Number

When the sum of all the factors (including 1 but excluding the number itself) of the given number is the same number then this number is called as Perfect Number.

For example 6, 28, 496, 8128, ...etc.

So far only 27 perfect numbers are known. The factors of 28 are 1, 2, 4, 7, 14, 28

Now, $1 + 2 + 4 + 7 + 14 = 28$

Hence 28 is a perfect Number.

Triangular Number

A triangular number is obtained by adding the previous number to the nth position in the sequence of triangular numbers, where the first triangular number is 1. The sequence of triangular numbers is given as follows

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, ...etc.

Introductory Exercise 1.3

1. The set of natural numbers is closed under the binary operations of :
(a) addition, subtraction, multiplication and division
(b) addition, subtraction, multiplication but not division
(c) addition and multiplication but not subtraction and division
(d) addition and subtraction but not multiplication and division

2. If p be a prime, $p > 3$ and let x be the product of natural numbers 1, 2, 3, ..., $(p-1)$, then consider the following statements :
1. x is a composite number divisible by p.
2. x is a composite number not divisible by p, but some prime greater than p may divide x.
3. x is not divisible by any prime $(p-2)$.
4. All primes less than $(p-1)$ divide x.

Of these statements :
(a) 1 and 2 are correct (b) 2 and 3 are correct
(c) 3 and 4 are correct (d) 4 alone is correct

3. Consider the following statements :
1. If $p > 2$ is a prime, then it can be written as $4n + 1$ or $4n + 3$ for suitable n.
2. If $p > 2$ is a prime, then $(p-1)(p+1)$ is always divisible by 4.

Of these statements :
(a) 1 and 2 are false (b) 1 and 2 are true
(c) 1 is true but 2 is false (d) 1 is false but 2 is true

4. A number n is said to be 'perfect' if the sum of all its divisors excluding n 'itself' is equal to n. An example of perfect number is :
(a) 9 (b) 15
(c) 21 (d) 6

5. If $2^p + 1$ is a prime number, then p must be power of

(a) 2 (b) 3
(c) 5 (d) 6

6. The number of composite numbers between 101 and 120 is :

(a) 11 (b) 12
(c) 13 (d) 14

7. The total number of prime numbers between 120 and 140 is :

(a) 7 (b) 6 (c) 5 (d) 4

8. The unit digit of every prime number (other than 2 and 5) must be necessarily :

(a) 1, 3 or 5 only (b) 1, 3, 7 or 9
(c) 7 or 9 only (d) 1 or 7 only

1.4 Integers

The extended set of whole numbers in which negative integers are also included is known as the set of integers and is denoted by

$$Z \text{ or } I = \{\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \ldots\}$$

(a) Positive integers : The set of integers $\{1, 2, 3, \ldots\}$ is known as positive integers.

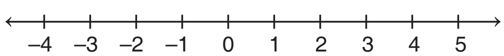
(b) Negative integers : The set of integers $\{-1, -2, -3, -4, \ldots\}$ is known as negative integers.

(c) Non-negative integers : The set of integers $\{0, 1, 2, 3, 3 \ldots\}$ is called as non-negative integers.

(d) Non-positive integers : The set of integers $\{0, -1, -2, -3, \ldots\}$ is called as non-positive integers.

- '0' is neither positive nor negative integer.

Representation of the Integers on a Number Line



All the integers (whole number, natural number etc.) can be shown on this number line, where every integer is represented by some point on the line.

It can be said that :

(a) There is no any largest or smallest integer.

(b) Every integer has a predecessor and a successor.

(c) An integer is smaller than all those integers which are on the right side of it and is greater than all those integers which are on the left side of it on the number line.

e.g., $-4 > -5, 3 > 1, 10 > -4$ etc.

or $-6 < 0, -2 < 1$ etc.

Exp. 1) Arrange the following integers in ascending order $-3, -7, 8, 5, 0, 3, 17, -23$.

Solution $-23, -7, -3, 0, 3, 5, 8, 17$

Exp. 2) Arrange the following integers in descending order $-17, 18, 15, 32, 81, -5, 87$.

Solution $87, 81, 32, 18, 15, -5, -17$

Properties of Integers

1. Closure law is followed by all the integers

2. Commutative law and Associative law is not followed by all the integers for the subtraction.

e.g., $4 - 6 \neq 6 - 4$

and $(4-6) - 2 \neq 4 - (6-2)$

but it is valid for the addition and multiplication as

$4 + 6 = 6 + 4$

or $(-3) + (-2) = (-2) + (-3)$

and $4 \times 6 = 6 \times 4$

or $-3 \times -2 = -2 \times -3$

and $(-2) + [(-3) + (-7)] = [(-2) + (-3)] + (-7)$ etc.

3. Additive identity of all the integers is zero (0) and Multiplicative identity of all the integers is 1.

e.g., $-3 + 0 = -3, \quad 8 + 0 = 8$

and $-5 \times 1 = -5, \quad 7 \times 1 = 7$ etc.

4. Additive inverse of an integer a is $-a$.

e.g. the additive inverse of 7, 8, 9, 15, – 3, – 5, – 6 etc. are – 7, – 8, – 9, – 15, 3, 5 and 6 respectively.

5. Distributive law of multiplication over addition or subtraction

i.e., $a \times (b \pm c) = a \times b \pm a \times c$

For example: $3 \times (4 \pm 6) = 3 \times 4 \pm 3 \times 6$

NOTE

1. Division by zero is not defined in Mathematics.
2. Division by 1 is actually unification (not division) so it is an improper divisor.

Some important rules regarding the sign convention in mathematical operations

(i) $(a) + (b) = +(a + b)$

$$(-a) + (b) = b - a$$
$$(a) + (-b) = a - b$$
$$(-a) + (-b) = -(a + b)$$

i.e., $(+) + (+) = +$

$(-)+(+)=+$ if the numerical value of $+$ is greater
$(-)+(+)=-$ if the numerical value of $-$ is greater
$(-)+(-)=-$

For example $(4)+(7)=11$, $(-3)+(8)=5$

$(-8)+(3)=-5$

$(-5)+(-3)=-8$ etc.

Similarly, $(5)-(2)=3$, $(2)-(5)=-3$

$(-5)-(2)=-7$, $(5)-(-2)=7$

$(-5)-(-2)=-3$, $(-2)-(-5)=3$

(ii) $a\times b=ab$ *i.e.,* $(+)\times(+)=+$

$-a\times b=-ab$ *i.e.,* $(-)\times(+)=-$

$a\times -b=-ab$ *i.e.,* $(+)\times(-)=-$

$-a\times -b=ab$ *i.e.,* $(-)\times(-)=+$

For example, $(2)\times(3)=6$, $(-2)\times(3)=-6$, $2\times(-3)=-6$ and $(-2)\times(-3)=6$

Similarly, $\frac{8}{2}=4$, $\frac{-8}{2}=-4$,

$\frac{8}{-2}=-4$, $\frac{-8}{-2}=4$

i.e., $\frac{(+)}{(+)}=(+)$, $\frac{(-)}{(+)}=(-)$,

$\frac{(+)}{(-)}=(-)$, $\frac{(-)}{(-)}=(+)$

Practice Exercise

Evaluate the following

1. 4 + (– 5) **2.** – 4 –(– 2)

3. 8 – (– 5) **4.** – 13 + (56)

5. (– 94) + (– 239) **6.** (– 526) – (– 217)

7. 7 + (– 5) + (– 2) **8.** – 6 + (– 2) – (– 3) + 1

9. 6 × 9 **10.** (– 9) × (– 13)

11. – 7 × 8 **12.** 13 × (– 15)

13. (– 3) × (8) × (– 5) **14.** (– 8) × (– 19) × (– 15)

15. [(– 6) × (– 8)] × 5 **16.** (– 18) ÷ 6

17. 126 ÷ (– 14) **18.** – 13 × (7 – 8)

19. (– 32) ÷ (– 4) **20.** (– 31) + (31)

Answers

1. –1	**2.** –2	**3.** 13	**4.** 43	**5.** –333
6. –309	**7.** 0	**8.** – 4	**9.** 54	**10.** 117
11. –56	**12.** –195	**13.** 120	**14.** –2280	**15.** 240
16. –3	**17.** –9	**18.** 13	**19.** 8	**20.** 0

Numerical Expression

Collection of numbers connected by one or more operations of addition, subtraction, multiplication and division is called a numerical expression.

It can also involve some brackets.

e.g., $7+18\div3\div1\times6$

and $84-6\div2-3\times(-7)$

and $9+(-2)\times\{72\div8\}+21$, etc.

Simplification Rules

The order in which various mathematical operations must be done can be remembered with the word 'BODMAS'

Where B $\rightarrow$ Brackets
O $\rightarrow$ Off
D $\rightarrow$ Division
M $\rightarrow$ Multiplication
A $\rightarrow$ Addition
S $\rightarrow$ Subtraction

So first of all we solve the innermost brackets moving outwards. Then we perform 'of ' which means multiplication then, Division, Addition and Subtraction.

- Addition and Subtraction can be done together or separately as required.
- Between any two brackets if there is no any sign of '+' or '–' it means we have to do multiplication *e.g.,*

$$(7)\ (2)=7\times2=14$$
$$[3(5)+7]=15+7=22$$

Brackets : They are used for the grouping of things or entities. The various kind of brackets are :

(i) '–' is known as line (or bar) bracket or vinculum
(ii) ( ) is known as parenthesis or common bracket
(iii) { } is known as curly bracket or brace.
(iv) [ ] is known as rectangular (or big) bracket.

The order of eliminating brackets is :

(i) line bracket
(ii) common bracket
(iii) curly bracket
(iv) rectangular bracket

The significance of various brackets is as follows:

{} contains only the particular values that are explicitly mentioned.

() contains all the values of the defined range, except the extreme values of the given range.

[] contains all the values of the defined range including both the extreme values of the given range.

(] contains all the values of the defined range, but does not include the lowest extreme value of the given range.

[) contains all the values of the defined range, but does not include the highest extreme value of the given range.

Exp. 1) Solve the following expressions :

(a) $3 + 2 - 1 \times 4 \div 2$
(b) $7 \times 3 + 8 - 2$
(c) $53 \times 2 - 1 \times 6$
(d) $12 + (-3) + 5 - (-2)$

Solution (a) $3 + 2 - 1 \times 4 \div 2 = 3 + 2 - 1 \times 2 = 3 + 2 - 2 = 3$
(b) $7 \times 3 + 8 - 2 = 21 + 8 - 2 = 27$
(c) $53 \times 2 - 1 \times 6 = 106 - 6 = 100$
(d) $12 + (-3) + 5 - (-2) = 12 - 3 + 5 + 2 = 9 + 7 = 16$

Exp. 2) Evaluate or simplify the following expression

(a) $9 - \{7 - 24 \div (8 + 6 \times 2 - 16)\}$
(b) $(-3) \times (-12) \div (-4) + 3 \times 6$
(c) $17 - \{8 \div (2 \times 3 - 4)\}$
(d) $5 \times 2 - [3 - \{5 - (7 + 2 \text{ of } 4 - 19)\}]$

Solution (a) $9 - \{7 - 24 \div (8 + 6 \times 2 - 16)\}$

$= 9 - \{7 - 24 \div (8 + 12 - 16)\} = 9 - \{7 - 24 \div 4\}$
$= 9 - \{7 - 6\} = 9 - \{1\} = 8$

(b) $(-3) \times (-12) \div (-4) + 3 \times 6 = 36 \div (-4) + 18$
$= -9 + 18 = 9$

(c) $17 - \{8 \div (2 \times 3 - 4)\} = 17 - \{8 \div (6 - 4)\}$
$= 17 - \{8 \div 2\} = 17 - \{4\} = 13$

(d) $5 \times 2 - [3 - \{5 - (7 + 2 \text{ of } 4 - 19)\}]$
$= 5 \times 2 - [3 - \{5 - (7 + 8 - 19)\}]$
$= 10 - [3 - \{5 - (-4)\}] = 10 - [3 - \{9\}]$
$= 10 - [-6] = 10 + 6 = 16$

Exp. 3) Which one of the following options is correct for each of the given expressions?

(i) $45 - [28 - \{37 - (15 - k)\}] = 58$, *the value of k is :*

(a) 19 (b) – 19
(c) – 39 (d) none of these

(ii) $1 + [1 \div \{5 \div 4 - 1 \div (13 \div 3 - 1 \div 3)\}]$ *is equal to :*

(a) 2/5 (b) 2
(c) 3/2 (d) none of these

(iii) $2 - [3 - \{6 - (5 \div \overline{4 - 3})\}]$ *is equal to :*

(a) 0 (b) 1
(c) – 3 (d) none of these

Ans. (i) (a), (ii) (b), (iii) (a)

Exp. 4) A student gets 4 marks for a correct answer and 1 mark is deducted for a wrong answer. If she has attempted 80 questions at all and she has got only 240 marks, the number of correct answers she has attempted is.

(a) 40 (b) 60
(c) 64 (d) can't be determined

Solution If she has attempted all the 80 questions correctly she must have got 320 (= 80 × 4) marks, but if she attempts one wrong answer it means she is liable to lose 5 marks (4 + 1). Thus for every wrong answer she loses 5 marks.

Now since she has lost total 80 marks (320 – 240 = 80).

This implies that she has attempted 16 questions wrong $\left(\because \frac{80}{5} = 16\right)$. It means she has done only 64 questions (80 – 16 = 64) correctly.

Hence (c) is the correct option.

Alternatively Since she has obtained less marks than the maximum possible marks it means she must have attempted some questions incorrectly.

Now, we can check through options. Let us consider option (c).

Correct answer	Wrong answers	Total questions
64	16	80
Marks for correct answers	Marks for wrong answers	Net marks
$64 \times 4 = 256$	$16 \times -1 = -16$	$256 - 16 = 240$

Hence the assumed option (c) is correct.

Alternatively If she has attempted x question correctly it means she has attempted $(80 - x)$ questions incorrectly.

So $4 \times (x) - 1 \times (80 - x) = 240$
$\Rightarrow$ $5x - 80 = 240$
$\Rightarrow$ $5x = 320$
$\Rightarrow$ $x = 64$

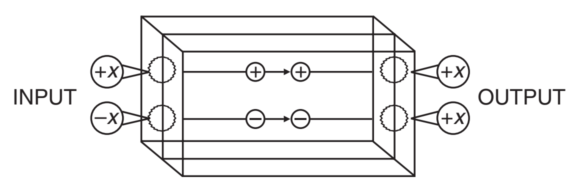
Hence she has attempted 64 answers correctly.

Absolute value of an integer or Modulus

The absolute value of an integer is its numerical value irrespective of its sign (or nature). The absolute value of an integer x is written as $|x|$ and is defined as

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Modulus (Machine Model)



It means any integer whether it is positive or negative if it is operated upon modulus, it always gives a positive integer.

for example $|-7| = 7$, $|-4| = 4$, $|3| = 3$ etc.

Also, $\text{Max } \{x, -x\} = |x|$

and $-\text{Min } \{x, -x\} = |x|$

and $\sqrt{(x)^2} = |x|$

Properties of a Modulus or Mod

1. $|a| = |-a|$
2. $|ab| = |a||b|$
3. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$
4. $|a+b| \le |a| + |b|$

   (The sign of equality holds only when the sign of a and b are same)
5. If $|a| \le k \Rightarrow -k \le a \le k$
6. If $|a-b| \le k \Rightarrow -k \le a-b \le k$
   $\Rightarrow b-k \le a \le b+k$

Exp. 1) Solution set of the equation $|x-2| = 5$ is :

(a) $\{3, -7\}$ (b) $\{-3, 7\}$
(c) $\{3, 6\}$ (d) none of these

Solution $|x-2| = 5 \Rightarrow x-2 = 5$

or $x - 2 = -5$

$\Rightarrow$ $x = 7$ or $x = -3$

Hence $x = \{-3, 7\}$, (b) is the correct option.

Exp. 2) The maximum value of the expression $27 - |9x-8|$ is

(a) 27 (b) 17
(c) 44 (d) 26

Solution The maximum value of the expression $27 - |9x-8|$ will be maximum only when the value of $|9x-8|$ is minimum, but the minimum possible value of any $|k|$ is zero. Hence the maximum value of $27 - |9x-8| = 27 - (0) = 27$

Hence option (a) is correct.

Exp. 3) The minimum value of the expression $|17x-8| - 9$ is

(a) 0 (b) -9
(c) $\dfrac{8}{17}$ (d) none of the above

Solution The minimum value of expression $|17x-8| - 9$ is minimum only when $|17x-8|$ is minimum. But the minimum value of $|k|$ is zero.

Hence minimum value of $|17x-8| - 9 = 0 - 9 = -9$

Exp. 4) If $2a - 9 = b + a$, then the value of $(|a-b| + |b-a|)$ is :

(a) 18 (b) 11
(c) 1 (d) 0

Solution $\because$ $2a - 9 = b + a8$

$\Rightarrow$ $2a - a = b + 9$

$\Rightarrow$ $a = b + 9$

$\Rightarrow$ $a - b = 9$ or $b - a = -9$

Hence $|a-b| + |b-a| = |9| + |-9|$
$= 9 + 9 = 18$

Thus (a) is the correct option.

Exp. 5) The value of x for which the value of $|3x+15|$ is minimum :

(a) 3 (b) 5
(c) -5 (d) none of the above

Solution The minimum value of $|3x+15| = 0$

$\Rightarrow$ $3x + 15 = 0 \Rightarrow x = -5$

Hence (c) is correct option.

Greatest Integer Less Than Or Equal To A Number

This number is represented as $\lfloor N \rfloor$. The resultant number is always an integral number.

1. when the given number is not an integer, the resultant number is the greatest integer less than the given number. For example, $\lfloor 3.1 \rfloor = 3, \lfloor 2.5 \rfloor = 2, \lfloor 1.99 \rfloor = 1, \lfloor 0.75 \rfloor = 0,$ $\lfloor -0.45 \rfloor = -1, \lfloor -1.45 \rfloor = -2$, etc.
2. When the given number is integer, the resultant number is the same as the given Number.

   For example, $\lfloor 3 \rfloor = 3, \lfloor 2 \rfloor = 2, \lfloor 0 \rfloor = 0,$
   $\lfloor -1 \rfloor = -1, \lfloor -2 \rfloor = -2$, etc

Least Integer Greater Than Or Equal To Number

This number is represented as $\lceil N \rceil$. The resultant number is always an integral number.

1. When the given number is not a integer, the resultant number is the least integer less than the given number. For example, $\lceil 3.1 \rceil = 4, \lceil 2.5 \rceil = 3, \lceil 1.99 \rceil = 2, \lceil 0.75 \rceil = 1,$ $\lceil -0.45 \rceil = 0, \lceil -1.45 \rceil = -1$, etc.
2. When the given number is integer, the resultant number is the same as the given Number. For example, $\lceil 3 \rceil = 3, \lceil 2 \rceil = 2, \lceil 0 \rceil = 0, \lceil -1 \rceil = -1, \lceil -2 \rceil = -2$, etc.

Introductory Exercise 1.4

1. $\frac{3}{4} \times \left(\frac{-2}{3} + \frac{3}{5}\right)$ is equal to :
   (a) $\frac{-3}{20}$ (b) $\frac{-19}{20}$
   (c) $\frac{-1}{20}$ (d) $\frac{1}{20}$
2. The value of $4 \times 100 + 3 \times 10 + \frac{9}{1000}$ is :
   (a) 430.09 (b) 430.0009
   (c) 430.009 (d) 430.900
3. If x and y are positive real numbers, then :
   (a) $x > y \Rightarrow -x > -y$ (b) $x > y \Rightarrow -x < -y$
   (c) $x > y \Rightarrow \frac{1}{x} > \frac{1}{y}$ (d) $x > y \Rightarrow \frac{-1}{x} < \frac{-1}{y}$
4. If $x < 0 < y$, then :
   (a) $\frac{1}{x^2} < \frac{1}{xy} < \frac{1}{y^2}$ (b) $\frac{1}{x^2} > \frac{1}{xy} > \frac{1}{y^2}$
   (c) $\frac{1}{x} < \frac{1}{y}$ (d) $\frac{1}{x} > 1$
5. On the set of integers I, if a binary operation 'o' be defined as $aob = a - b$ for every $a, b \in I$, then :
   (a) association law holds
   (b) commutative law holds
   (c) I is not closed under this operation
   (d) I is closed under this operation
6. If $|x - 2| < 3$ then :
   (a) $1 < x < 5$
   (b) $-1 < x < 2$
   (c) $-1 < x < 5$
   (d) $0 < x < 6$
7. How many of the following relations is/are always true for any real values of p and q?
   (i) $|p + q| = |p| + |q|$ (ii) $|p - q| = |p| - |q|$
   (iii) $\left|\frac{p}{q}\right| = \frac{|p|}{|q|}$ (iv) $|pq| = |p|\,|q|$
   (v) $|p^q| = |p|^{|q|}$ (vi) $|\sqrt[q]{p}| = \sqrt[|q|]{|p|}$
   (a) None (b) one
   (c) two (d) three
8. Which of the following relations is NOT always true?
   (a) $|p + q| \le |p| + |q|$
   (b) $|p - q| \le |p| - |q|$
   (c) $|p| - |q| > 0$, *if* $p^2 - q^2 > 0$
   (d) $|p - q|^2 = (p - q)^2$
9. The total number of integral values of x for which $|||x - 2| - 4| - 6| < 10$
   (a) 39 (b) 42
   (c) 43 (d) 38
10. If x satisfies $|x - 1| + |x - 2| + |x - 3| \ge 6$, then :
    (a) $0 \le x \le 4$
    (b) $x \le 0$ or $x \ge 4$
    (c) $x \le -2$ or $x \ge 4$
    (d) $x \ge -2$ or $x \le 4$
11. For the positive integers a b, c, d find the minimum value of $(-1)^a + (-1)^b + (-1)^c + (-1)^d$, If $a + b + c + d = 1947$.
    (a) -1 (b) -2
    (c) -3 (d) -4

Directions (for Q. Nos. 12 and 13) *In a test, conducted by Lamamia, a student attempted n problems out of total 30 problems and she scored net 17 marks. In this test each correct response awards 3 marks and each wrong response penalizes the test taker by deducting 1 mark from the total score.*

12. At most how many problems are attempted by the student?
    (a) 7
    (b) 10
    (c) 17
    (d) none of the above
13. If there are total m students who took the same test in which all the students scored net 17 marks and none of these students attempted same number of problems, what is the maximum value of m?
    (a) 17 (b) 10
    (c) 7
    (d) none of the above

1.5 Factors and Multiples

Product

When two or more number are multiplied together then the resultant value is called the product of these numbers.

For example

$$3\times4=12,\quad 2\times7\times5=70,\quad 3\times5\times11=165$$

where 12, 70 and 165 are called the products.

But we see that $12=4\times3$ it means 4 and 3 are the **factors of** 12.

and $\qquad 70=2\times5\times7$

where 2, 5, 7 are known as **factors** of 70.

Again 12 is called as the **multiple of** 3 or **multiple** of 4.

Similarly 70 is called as the **multiple of** 2 or 5 or 7 or 10 or 35 or 14.

Thus a number which divides a given number exactly is called factor (or divisor) of that given number and the given number is called a multiple of that number.

Now, 15 is exactly divisible by 1, 3, 5, 15 so 1, 3, 5, 15 are called as the factors of 15 while 15 is called as the multiple of these factors. Where 1, 15 are **improper factors** and 3, 5 are called **proper factors of** 15.

Therefore 1 and itself (the number) are called the improper factors of the given number.

So the factors of $24=1, 2, 3, 4, 6, 8, 12, 24$

Factors of $35=1, 5, 7, 35$

Factors of $36=1, 2, 3, 4, 6, 9, 12, 18, 36$

Similarly the multiples of 2 are 2, 4, 6, 8, 10, 12, ...

multiples of 7 are 7, 14, 21, 28, 35, 42, 49....

multiples of 10 are 10, 20, 30, 40, 50...

- 1 is a factor of every number.
- Every number is a factor of itself.
- Every number, except 1 has atleast two factors *viz.,* 1 and itself.
- Every factor of a number is less than or equal to that number.
- Every multiple of a number is greater than or equal to itself.
- Every number has infinite number of its multiples.
- Every number is a multiple of itself.

Prime Factorisation

If a natural number is expressed as the product of prime numbers (factors) then the factorisation of the number is called its prime factorisation.

For example :

(i) $72=2\times2\times2\times3\times3$

$$\begin{array}{r|r} 2 & 72 \\ \hline 2 & 36 \\ \hline 2 & 18 \\ \hline 3 & 9 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

(ii) $420=2\times2\times3\times5\times7$

$$\begin{array}{r|r} 2 & 420 \\ \hline 2 & 210 \\ \hline 3 & 105 \\ \hline 5 & 35 \\ \hline 7 & 7 \\ \hline & 1 \end{array}$$

(iii) $3432=2\times2\times2\times3\times11\times13$

$$\begin{array}{r|r} 2 & 3432 \\ \hline 2 & 1716 \\ \hline 2 & 858 \\ \hline 3 & 429 \\ \hline 11 & 143 \\ \hline 13 & 13 \\ \hline & 1 \end{array}$$

Practice Exercise

Direction (for Q. Nos 1-6) *factorize the following numbers into the prime factors.*

1. 210 **2.** 120 **3.** 3315

4. 3465 **5.** 1197 **6.** 157573

7. Find the least number by which 22932 must be multiplied or divided so as to make it a perfect square:
(a) 10 (b) 4
(c) 11 (d) 13

8. How many natural numbers upto 150 are divisible by 7?
(a) 21 (b) 14
(c) 22 (d) 17

9. How many numbers between 333 and 666 are divisible by 5?
(a) 67 (b) 70
(c) 75 (d) 55

10. How many numbers between 11 and 111 are the multiples of both 2 and 5?
(a) 10 (b) 12
(c) 11 (d) 70

Answers & Solutions

1. $2\times3\times5\times7$ **2.** $2\times2\times2\times3\times5$

3. $3\times5\times13\times17$ **4.** $3\times3\times5\times7\times11$

5. $3\times3\times7\times19$ **6.** $13\times17\times23\times31$

7. $22932=2^2\times3^2\times7^2\times13$. Therefore in order to make 22932 a perfect square you can either divide or multiply by 13.

8. 7, 14, 21, 28, 35, ...,140, 147.
7 (1, 2, 3, ...20, 21)
Therefore we have 21 natural numbers between 1 and 150 which are divisible by 7.

9. 335, 340,....., 660, 665
5(67, 68,... , 132, 133)
Therefore the required numbers = 133 – 66 = 67.

10. 20, 30, 100, 110.
10(2, 3, , 11)
Therefore the required number $= 11 - 1 = 10$

Number of Prime Factors of a Composite Number

Let us assume a' composite number say 24, then find the number of prime factors of 24.

$$24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3^1$$

We can see that there are two unique prime factors 2 and 3, however, the power of 2 is 3 and the power of 3 is 1, so the total number of prime factors is $4 (= 3 + 1)$.

Let there be a composite number N and its prime factors be a, b, c, d, ... etc. and p,q,r,s, ... be the indices (or powers) of $a,b.c,d$, ... then the number of total prime factors (or divisors) of N

$$= p + q + r + s + ...$$

Exp. 1) Find the number of unique Prime factors of 210.

Solution $210 = 2 \times 3 \times 5 \times 7$

Therefore 210 has 4 unique prime factors, namely, 2, 3, 5 and 7.

Exp. 2) Find the total number of Prime factors of 210

Solution $210 = 2 \times 3 \times 5 \times 7 = 2^1 \times 3^1 \times 5^1 \times 7^1$

Therefore the total number of prime factors of

$$210 = 1 + 1 + 1 + 1 = 4$$

Exp. 3) Find the number of unique Prime factors of 120.

Solution $120 = 2x2x2x3x5 = 2^3 \times 3^1 \times 5^1$

Therefore 120 has 3 unique prime factors, namely 2, 3 and 5.

Exp. 4) Find the total number of Prime factors of 120

Solution $120 = 2 \times 2 \times 2 \times 3 \times 5 = 2^3 \times 3^1 \times 5^1$

Therefore the total number of prime factors of

$$120 = 3 + 1 + 1 = 5$$

Exp. 5) Find the number of unique Prime factors of 600.

Solution $600 = 2 \times 2 \times 2 \times 3 \times 5 \times 5 = 2^3 \times 3^1 \times 5^2$

Therefore 600 has 3 unique prime factors, namely, 2, 3 and 5.

Exp. 6) Find the total number of Prime factors of 600

Solution $600 = 2 \times 2 \times 2 \times 3 \times 5 \times 5 = 2^3 \times 3^1 \times 5^2$

Therefore the total number of prime factors of $600 = 3 + 1 + 2 = 6$

Number of All the Factors of a Composite Number

Number of Factors (or Divisors) of A Given Number (composite number)

Let us assume a composite number say 24 then find the no. of factors.

$$24 = 1 \times 24$$
$$2 \times 12$$
$$3 \times 8$$
$$4 \times 6$$

We see that there are total 8 factors namely,

1, 2, 3, 4, 6, 8, 12 and 24. But for the larger numbers it becomes difficult to find the total number of factors. So we use the following formula which can be derived with the help of theory of permutation and combination. Let there be a composite number N and its prime factors be $a, b, c, d, \ldots$ etc. and $p, q, r, s, \ldots$ etc, be the indices (or powers) of the $a, b, c, d, \ldots$ etc, respectively. That is if N can be expressed as $N = a^p . b^q . c^r . d^s ...$, then the number of total divisors or factors of N is $(p+1)(q+1)(r+1)(s+1)\ldots$

Exp. 1) Find the total number of factors of 24.

Solution $24 = 2^3 \times 3^1$ $[\because 24 = 2 \times 2 \times 2 \times 3]$

$\therefore$ Number of factors $= (3 + 1)(1 + 1) = 8$

Exp. 2) Find the total number of factors of 540.

(a) 24 (b) 20 (c) 30 (d) none

Solution $540 = 2 \times 2 \times 3 \times 3 \times 3 \times 5$ or $540 = 2^2 \times 3^3 \times 5^1$

Therefore total number of factors of 540 is

$$(2 + 1)(3 + 1)(1 + 1) = 24$$

Exp. 3) The total number of divisors of 10500 except 1 and itself is :

(a) 48 (b) 50 (c) 46 (d) 56

Solution $10500 = 2 \times 2 \times 3 \times 5 \times 5 \times 5 \times 7$

$$10500 = 2^2 \times 3^1 \times 5^3 \times 7^1$$

$\therefore$ Total number of divisors of 10500 is

$$(2 + 1)(1 + 1)(3 + 1)(1 + 1) = 48$$

But we have to exclude 1 and 10500

So there are only $48 - 2 = 46$ factors of 10500 except 1 and 10500.

Hence, (c) is the correct option.

Exp. 4) Total number of factors of 36 is :

(a) 4 (b) 9 (c) 6 (d) none

Solution $36 = 2^2 \times 3^2$

$\therefore$ Number of factors $= (2 + 1)(2 + 1) = 3 \times 3 = 9$

Hence (b) is the correct option.

NOTE A perfect square number always contains odd no. of factors.

Practice Exercise

Find the number of factors of the following numbers :

1. 1008 **2.** 101 **3.** 111 **4.** 7056 **5.** 18522
6. 7744 **7.** 3875 **8.** 1458 **9.** 1339 **10.** 512

Answers

1. 30	**2.** 2	**3.** 4	**4.** 45	**5.** 32
6. 21	**7.** 8	**8.** 14	**9.** 4	**10.** 10

Number of Odd Factors of a Composite Number

Let us assume as small number, e.g. 24, which can be expressed as $24 = 2^3 \times 3^1$

Now, we have $24 = (1 \times 24) = (2 \times 12) = (3 \times 8)$

Now, we can see that there are total 2 odd factors, namely, 1 and 3.

Further, assume another number, say, 36 which can be expressed as $36 = 2^2 \times 3^2$

Now, we have $36 = (1 \times 36) = (2 \times 18) = (3 \times 12)$
$= (4 \times 9) = (6 \times 6)$

So, we can see that there are only 3 odd factors, namely, 1, 3 and 9.

Once again we assume another number, say, 90, which can be expressed as $90 = 2 \times 3^2 \times 5$.

Now, we have $90 = (1 \times 90) = (2 \times 45) = (3 \times 30)$
$= (5 \times 18) = (6 \times 15) = (9 \times 10)$

Thus, there are only 6 odd factors, namely, 1, 3, 5, 9, 15, 45.

To get the number of odd factors of a number N first of all express the number N as $N = (p_1{}^a \times p_2{}^b \times p_3{}^c \times \ldots) \times (e^x)$

(where, $p_1, p_2, p_3, \ldots$ are the odd prime factors and e is the even prime factor.)

Then the total number of odd factors

$$= (a+1)(b+1)(c+1)\ldots$$

Exp. 1) The number of odd factors (or divisors) of 24 is:

Solution $\because$ $24 = 2^3 \times 3^1$

Here 3 is the odd prime factor

So, total number of odd factors = (1 + 1) = 2

Exp. 2) The number of odd factors of 36 is :

Solution $\because$ $36 = 2^2 \times 3^2$

$\therefore$ Number of odd factors = (2 + 1) = 3

Exp. 3) The number of odd factors of 90 is ....

Solution $\because$ $90 = 2^1 \times 3^2 \times 5^1$

$\therefore$ Total number of odd factors of

$$90 = (2+1)(1+1) = 6$$

Number of Even Factors of a Composite Number

Number of even factors of a number

= (Total number of factors of the given number – Total number of odd factors)

Thus in the above

Example (1), number of even factors $= 8 - 2 = 6$

Example (2), number of even factors $= 9 - 3 = 6$

Example (3), number of even factors $= 12 - 6 = 6$

Sum of Factors of a Composite Number

Once again if you want to find the sum of a small composite number, then you can do it manually, but for larger number it is a problem.

e.g., Sum of factors of $24 = 1 + 2 + 3 + 4 + 6 + 8 + 12 + 24 = 60$

Let N be the composite number and $a, b, c, d..$ be its prime factors and p, q, r, s be the indices (or powers) of a, b, c, d. That is if N can be expressed as

$$N = a^p \,.\, b^q \,.\, c^r \,.\, d^s \ldots$$

then the sum of all the divisors (or factors) of N

$$= \frac{(a^{p+1} - 1)(b^{q+1} - 1)(c^{r+1} - 1)(d^{s+1} - 1)}{(a-1)(b-1)(c-1)(d-1)} \ldots$$

Exp. 1) Find the sum of factors of 24.

Solution $\because$ $24 = 2^3 \times 3^1$

$$\therefore \text{ Sum of factors of } 24 = \frac{(2^4 - 1)(3^2 - 1)}{(2-1)(3-1)} = \frac{15 \times 8}{1 \times 2} = 60$$

Exp. 2) Find the sum of factors of 270.

Solution $\because$ $270 = 2^1 \times 3^3 \times 5^1$

$$\therefore \text{ Sum of factors of } 270 = \frac{(2^{1+1} - 1)(3^{3+1} - 1)(5^{1+1} - 1)}{(2-1)(3-1)(5-1)} = \frac{3 \times 80 \times 24}{1 \times 2 \times 4} = 720$$

Exp. 3) The sum of factors of 1520 except the unity is :

(a) 3720 (b) 2730
(c) 2370 (d) none of these

Solution Since $1520 = 2^4 \times 5^1 \times 19^1$

$\therefore$ Sum of all the factors of $1520 = \dfrac{(2^5 - 1)(5^2 - 1)(19^2 - 1)}{(2-1)(5-1))(19-1)}$

$$= \frac{31 \times 24 \times 360}{1 \times 4 \times 18} = 3720$$

But since unity is to be excluded.

$\therefore$ The net sum of the factors = 3719

$\therefore$ (d) is the correct option.

Exp. 4) The sum of factors of 19600 is :

(a) 54777 (b) 33667
(c) 5428 (d) none of these

Solution $19600 = 2^4 \times 5^2 \times 7^2$

$\therefore$ Sum of factors of $19600 = \dfrac{(2^{4+1} - 1)(5^{2+1} - 1)(7^{2+1} - 1)}{(2-1)(5-1)(7-1)}$

$$= \frac{31 \times 124 \times 342}{1 \times 4 \times 6} = 54777$$

Hence (a) is the correct option.

Product of Factors of a Composite number

Let us assume a very small number 24 and see the factors

$$24 = 1 \times 24 = 2 \times 12 = 3 \times 8 = 4 \times 6$$

Now, it is obvious from the above explanation that the product of factors of 24 $= (1 \times 24) \times (2 \times 12) \times (3 \times 8) \times (4 \times 6)$

$$= 24 \times 24 \times 24 \times 24 = (24)^4$$

Thus, the product of factors of composite number $N = N^{n/2}$, where n is the total number of factors of N.

Exp. 1) Product of divisors of 7056 is :

(a) $(84)^{48}$ (b) $(84)^{44}$
(c) $(84)^{45}$ (d) none of these

Solution $\because$ $7056 = 2^4 \times 3^2 \times 7^2$

$\therefore$ Number of factors/divisors of $7056 = (4+1)(2+1)(2+1) = 45$

$\therefore$ Product of factors $= (7056)^{45/2} = (84)^{45}$

Hence (c) is the correct option.

Exp. 2) Product of factors of 360 is :

(a) $(360)^{12}$ (b) $(36)^{120}$
(c) $(360)^{22}$ (d) $6^{24} \times 10^{10}$

Solution $\because$ $360 = 2^3 \times 3^2 \times 5^1$

$\therefore$ Number of factors of $360 = (3+1)(2+1)(1+1) = 24$

Thus the product of factors $= (360)^{24/2} = (360)^{12}$

Hence (a) is the correct option.

NOTE As we know the number of divisors of any perfect number is an odd number so we can express the given number in the form of of the perfect square to eliminate the $\frac{1}{2}$ from the power of the number as it can be seen in the above example no. 1.

Number of Ways of Expressing a Composite Number as a Product of two Factors

It has long been discussed in the previous examples of composite numbers but you might have not noticed it.

Let us consider an example of small composite number say, 24

Then
$$\begin{aligned} 24 &= 1 \times 24 \\ &= 2 \times 12 \\ &= 3 \times 8 \\ &= 4 \times 6 \end{aligned}$$

So it is clear that the number of ways of expressing a composite no. as a product of two factors

$$= \frac{1}{2} \times \text{the no. of total factors}$$

Exp. 1) Find the number of ways of expressing 180 as a product of two factors.

Solution $180 = 2^2 \times 3^2 \times 5^1$

Number of factors $= (2+1)(2+1)(1+1)$
$= 18$

Hence, there are total $\frac{18}{2} = 9$ ways in which 180 can be expressed as a product of two factors.

NOTE As you know when you express any perfect square number 'N' as a product of two factors namely $\sqrt{N}$ and $\sqrt{N}$, and you also know that since in this case $\sqrt{N}$ appears two times but it is considered only once while calculating the no.of factors so we get always an odd number as number of factors so we can not divide the odd number exactly by 2 as in the above formula. So if we have to consider these two same factors then we find the number of ways of expressing N as a product of two factors $= \dfrac{(\text{Number of factors} + 1)}{2}$.

Again if it is asked that find the no. of ways of expressing N as a product of Two distinct factors then we do not consider 1 way (*i.e.,* $N = \sqrt{N} \times \sqrt{N}$) then no. of ways $= \dfrac{(\text{Number of factors} - 1)}{2}$.

Exp. 2) Find the number of ways expressing 36 as a product of two factors.

Solution $36 = 2^2 \times 3^2$

Number of factors
$= (2+1)(2+1) = 9$

Hence the no. of ways of expressing 36 as a product of two factors $= \dfrac{(9+1)}{2} = 5$.

as $36 = (1 \times 36) = (2 \times 18) = (3 \times 12) = (4 \times 9) = (6 \times 6)$

Exp. 3) In how many ways can 576 be expressed as the product of two distinct factors?

Solution $\because$ $576 = 2^6 \times 3^2$

$\therefore$ Total number of factors

$$= (6 + 1)(2 + 1) = 21$$

So the number of ways of expressing 576 as a product of two distinct factors

$$= \frac{(21 - 1)}{2} = 10$$

NOTE Since the word 'distinct' has been used therefore we do not include 26 two times.

Number of Ways of Expressing a Composite Number as a Product of Three Factors

Exp. 1) Find the number of ways of expressing 2310 as product of 3 distinct factors.

Solution $a \times b \times c = 2310 = 2^1 \times 3^1 \times 5^1 \times 7^1 \times 11^1$

There are total 5 prime numbers. Since we know that 1 can also be a factor of 2310, so we have to consider 3 different cases.

Case I When out of three factors two factors are 1 and 1. That is $1 \times 1 \times 2310$

Thus we can express 2310 as a product of 3 factors in 1 way.

Case II When out of three factors one factor is 1. That is $(1 \times 2 \times 1150), (1 \times 3 \times 770), \ldots (1 \times 15 \times 77)$.

There are two possibilities to group the prime numbers: $(1 \times p \times pppp)$ or $(1 \times pp \times pp)$

This can be done in ${}^5C_1 + {}^5C_2 = 10 + 15 = 15$ ways.

Case III When none of the three factors is 1. That is $(2 \times 3 \times 385), (2 \times 5 \times 231), \ldots (2 \times 15 \times 77)$

There are two possibilities to group the prime numbers: $(p \times pp \times pp)$ or $(ppp \times p \times p)$. This can be done in

$${}^5C_1\left(\frac{{}^4C_2 \times {}^2C_2}{2}\right) + {}^5C_3\left(\frac{{}^2C_1 \times {}^1C_1}{2}\right) = 15 + 10 = 25 \text{ ways.}$$

Therefore the number of ways of expressing 2310 as a product of 3 factors

$$= 1 + 15 + 25 = 41$$

NOTE Refer the Permutation and Combination chapter to understand this topic better.

Alternatively If a number can be expressed as a product of n distinct prime numbers, it can be expressed as a product of 3 numbers in $\frac{3^{n-1} + 1}{2}$ ways.

Now $2310 = 2 \times 3 \times 5 \times 7 \times 11$. That is 2310 can be expressed as a product of 5 distinct prime numbers.

Therefore 2310 can be expressed as a product of 3 factors in

$$\frac{3^{n-1} + 1}{2} = \frac{3^4 + 1}{2} = 41 \text{ ways}$$

When the third factor is	The other two factors have prime numbers	The number of ways of expressing the other prime numbers into two factors	The number of ways of expressing 2310 as a product of 3 factors	The number of ways of expressing 2310 as a product of 3 Distinct factors
1	2, 3, 5, 7, 11	16	16	16 – 1 = 15
2	3, 5, 7, 11	8	8	8 – 1 = 7
3	2, 5, 7, 11	8	8	8 – 2 = 6
5	2, 3, 7, 11	8	8	8 – 3 = 5
7	2, 3, 5, 11	8	8	8 – 4 = 4
11	2, 3, 5, 7	8	8	8 – 5 = 3

Number of Co-Primes of A Composite Number N

For a composite number $N = a^p \times b^q \times c^r$ .., the number of co-primes of N is given by

$$\varphi(N) = N\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right)\ldots,$$

where a, b, c are prime numbers.

Exp. 1) Find the number of co-prime numbers of 10.

Solution $10 = 2 \times 5$ Number of co-prime numbers of 10 is given by $\varphi(10) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) = 4$. These numbers are 1, 3, 7, 9.

Exp. 2) Find the number of co-prime numbers of 336.

Solution $10 = 2^4 \times 3 \times 7$ Number of co-prime numbers of 336 is given by $\varphi(336) = 336\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{7}\right) = 96$

Exp. 3) Find the number of co-prime numbers of 49, which are composite numbers.

Solution $49 = 7^2$ Number of co-prime numbers of 49 is given by

$$\varphi(49) = (1 - 1/7) = 42$$

These numbers are 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 41, 43, 44, 45, 46, 47, 48.Since, there are 15 numbers, 1, 2, 3, 5, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47 which are not the composite numbers, so the required number = 42 – 15 = 27.

Sum of All The Co-Primes of a Composite Number N

For a composite number $N = a^p \times b^q \times c^r$ ...., the sum of all the co-primes of N is given by

$$\frac{N}{2}[\varphi(N)] = \frac{N}{2}\left[N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots\right],$$

where $a.b.c$ are prime numbers.

Exp. 1) Find the sum of all the co-prime numbers of 10.

Solution $10 = 2\times5$

Sum of all the co-prime numbers of 10 is given by

$$\frac{10}{2}[\varphi(10)] = \frac{10}{2}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right) = 20$$

The required sum is $1+3+7+9=20$

Exp. 2) Find the sum of all the co-prime numbers of 336.

Solution $336 = 2^4 \times 3 \times 7$

Sum of all the co-prime numbers of 336 is given by

$$\frac{336}{2}[\varphi(336)] = \frac{336}{2}\left[336\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{7}\right)\right] = 16128$$

Exp. 3) Find the sum of all the co-prime numbers of 49, which are composite number.

Solution $49 = 7^2$

Sum of all the co-prime numbers of 49 is given by

$$\frac{49}{2}[\varphi(49)] = \frac{49}{2}\left[49\left(1-\frac{1}{7}\right)\right] = 1029$$

The required sum

$$= 1029 - (1+2+3+5+11+13+17+19+23+29+31+37+41+43+47)$$
$$= 707.$$

Number of Ways of Expressing a Composite Number As A Product of Co-Prime Factors

For a composite number N, the number of ways of expressing a composite number as product of co-prime factors is 2^{n-1}, where n is the number of unique prime factors of N.

Exp. 1) Find the number of ways of expressing 10 as a product of co-prime factors.

Solution $10 = 2\times5$

The required number of ways $= 2^{(2-1)} = 2$

$(1\times10), (2\times5)$ are two different ways in which 10 can be expressed as a product of co-prime factors.

Exp. 2) Find the number of ways of expressing 336 as a product of co-prime factors.

Solution $336 = 2^4 \times 3 \times 7$

The required number of ways $= 2^{(3-1)} = 4$

$(1\times336), (16\times21), (48\times7), (112\times3)$ are four different ways of expressing 336 as a product of co-prime factors.

Exp. 3) Find the number of ways of expressing 49 as a product of co-prime factors.

Solution $49 = 7^2$

The required number of ways $= 2^{(1-1)} = 1$,

(1×49) is the only way to express 49 as a product of co-prime factors.

Introductory Exercise 1.5

1. The number of prime factors in the expression $6^4 \times 8^6 \times 10^8 \times 12^{10}$ is :
(a) 48 (b) 64 (c) 72 (d) 80

2. A number is expressed as $2^m \times 3^n$ and the sum of all its factors is 124, find m and n.
(a) 2, 3 (b) 3, 1 (c) 4, 1 (d) 3, 2

3. The sum of all possible factors of 500 (including 1 and 500 themselves) equals :
(a) 784 (b) 980 (c) 1092 (d) 1350

4. The perimeter of a rectangle is 72 cm. If the sides are positive integers, maximum how many distinct areas can it have?
(a) 18 (b) 9 (c) 36 (d) 35

5. The area of a rectangle is 72 sq cm. If the sides are positive integers, maximum how many distinct perimeters can it have?
(a) 12 (b) 8 (c) 9 (d) 6

6. Among the first 100 even natural numbers how many numbers have even number of factors?
(a) 37 (b) 50 (c) 73 (d) 93

7. In a set of first 350 natural numbers find the number of integers, which are not divisible by 5.
(a) 280 (b) 75 (c) 150 (d) 240

8. In a set of first 350 natural numbers find the number of integers, which are not divisible by 7.
(a) 120 (b) 280 (c) 300 (d) 240

9. In a set of first 350 natural numbers find the number of integers, which are not divisible by any of the numbers 5 or 7.
(a) 240 (b) 120
(c) 150 (d) 300

10. In a set of first 420 natural numbers find the number of integers that are divisible by 7 but not by any of 2, 3 or 5.
(a) 16 (b) 24
(c) 96 (d) 42

11. In a set of first 210 natural numbers find the number of integers which are not divisible by any of 2, 3, 4, ..., 10.
(a) 42 (b) 52
(c) 48 (d) 63

12. In a set of first 61 natural numbers find the number of integers, which are divisible by neither 2 nor 3 nor 5.
(a) 0
(b) 1
(c) 16
(d) 17

13. In a set of first 123 natural numbers find the number of integers, which are divisible by either 2 or 3 or 5.
(a) 90 (b) 31
(c) 33 (d) 11

14. In a set of first 180 natural numbers find the number of integers, which are divisible by neither 2 nor 3 nor 5 nor 7.
(a) 14 (b) 15 (c) 41 (d) 51

15. In a set of first 180 natural numbers find the number of prime numbers.
(a) 48 (b) 24 (c) 41 (d) 38

16. In a set of first 1000 natural numbers find the number of prime numbers.
(a) 181 (b) 168 (c) 200 (d) 224

1.6 HCF and LCM

In the previous articles I have discussed in detail about the factors. Now I will move towards common factor, Highest common factors (HCF) and least common multiple (LCM).

Common Factors

When any factor which is the factor of two or more given numbers then it is said that this particular factor is common.

For example $6 = 2 \times 3$

$15 = 3 \times 5$

We see that 3 is a common factor in both 6 and 15.

Similarly, $6 = 2 \times 3$

$8 = 2 \times 2 \times 2$

$30 = 2 \times 3 \times 5$

So, we can see that only 2 is common in all the three numbers 6, 8 and 30.

No other factor is common in all the three numbers.

Highest Common Factor (HCF) *or Greatest Common Divisor* (GCD)

HCF of two or more than two numbers is the greatest possible number that can divide all these numbers exactly, without leaving any remainder. For example, find the HCF of 84 and 126.

$\because \quad 84 = 2 \times 3 \times 7 \times 2 = 42 \times 2$

$126 = 2 \times 3 \times 7 \times 3 = 42 \times 3$

So, the HCF of 84 and 126 $= 2 \times 3 \times 7 = 42$

Thus we can see that 42 is the greatest common divisor since it can divide exactly both 84 and 126.

Basically there are two methods of finding the HCF.

(i) Factor Method
(ii) Division Method

(i) ***Factor Method***

In this method first we break (or resolve) the numbers into prime factors then take the product of all the common factors. This resultant product is known as the HCF of the given numbers.

Exp. 1) Find the HCF of 1680 and 3600.

Solution $\because$ $1680 = 2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 7$

$3600 = 2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 3 \times 5$

So the product of common factors $= 2 \times 2 \times 2 \times 2 \times 3 \times 5 = 240$

Hence, the HCF of 1680 and 3600 is 240.

Exp. 2) Find the HCF of 750, 6300, 18900.

Solution $750 = 2 \times 3 \times 5 \times 5 \times 5$

$6300 = 2 \times 3 \times 5 \times 5 \times 3 \times 2 \times 7$

$18900 = 2 \times 3 \times 5 \times 5 \times 3 \times 2 \times 3 \times 7$

So the product of common factors $= 2 \times 3 \times 5 \times 5 = 150$

Hence 150 is the HCF of 750, 6300, 18900.

(ii) ***HCF by Division Method***

Consider two smallest numbers, then divide the larger one of them by the smaller one and then divide this divisor by the remainder and again divide this remainder by the next remainder and so on until the remainder is zero. If there are more than two numbers of which HCF is to be found, we continue this process as we divide the third lowest number by the last divisor obtained in the above process.

Exp. 1) Find the HCF of 120 and 180.

Solution

120) 180 (1
120
60) 120 (2
120
×

Hence, 60 is the HCF of 120 and 180.

Exp. 2) Find the HCF of 420 and 1782.

Solution

```
420)1782(4
    1680
    ----
     102)420(4
         408
         ---
          12)102(8
              96
              --
               6)12(2
                 12
                 --
                  ×
```

Hence 6 is the HCF of 420 and 1782.

Exp. 3) Find the HCF of 210, 495 and 980.

Solution First we consider the two smallest numbers *i.e.,* 210 and 495.

```
210)495(2
    420
    ---
     75)210(2
        150
        ---
         60)75(1
            60
            --
            15)60(4
               60
               --
                ×

15)980(65
   975
   ---
    5)15(3
      15
      --
       ×
```

Hence the HCF is 5.

Euclidean Algorithm

If you watch carefully the above discussion you will find that the HCF is the factor of difference of the given numbers. So there is short cut for you to find the HCF.

You can divide the given numbers by their lowest possible difference if these numbers are divisible by this difference then this difference itself is the HCF of the given numbers. If this difference is not the HCF then any factor of this difference must be the HCF of the given numbers.

Be'zout's Identity

If 'H' be the HCF of any two positive integers a and b then there exists unique integers p and q such that $H = ap + bq$.

Exp. 1) Find the HCF of 63 and 84.

Solution The difference = 84 – 63 = 21

Now divide 63 and 84 by 21, since the given numbers 63 and 84 are divisible by their difference 21, thus 21 itself is the HCF of 63 and 84.

Exp. 2) Find the HCF of 42 and 105.

Solution The difference = 105 – 42 = 63

Now divide 42 and 105 by 63 but none of these is divisible by their difference 63.

So we factorize 63 and then divide 42 and 105 by the factor of 63. The greatest factor of 63 which can divide 42 and 105 that will be the HCF of the given numbers.

So $\quad 63 = 1 \times 63,\ 3 \times 21, 7 \times 9$

$\therefore$ The factor of 63 are 1, 3, 7, 9, 21 and 63.

Obviously we divide 42 and 105 by 21 (since the division by 63 is not possible) and we see that the given numbers are divisible by 21, hence 21 is the HCF of 42 and 105.

Exp. 3) Find the HCF of 30, 42 and 135.

Solution The difference between 30 and 42 = 12

The difference between 42 and 135 = 93

Now divide, 30, 42 and 135 by 12 but it can not divide all the numbers. So take the factors of 12.

The factors of 12 are 1, 2, 3, 4, 6, 12

Now we divide 30, 42 and 135 by 6 but it can not divide all these numbers.

So we divide now 30, 42 and 135 by 4, but 4 too can not divide all the numbers, Now we divide the given numbers by 3 and we will find that all the numbers are divisible by 3 hence 3 is the HCF of the given numbers.

HCF with Remainders

Case 1. Find the greatest possible number with which when we divide 37 and 58, it leaves the respective remainder of 2 and 3.

Solution Since when we divide 37 and 58 by the same number then we get remainders 2 and 3 respectively.

So $(37 - 2)$ and $(58 - 3)$ must be divisible, hence leaving the remainders zero. It means 35 and 55 both are divisible by that number so the HCF of 35 and 55 is 5.

Hence the greatest possible number is 5.

Case 2. Find the largest possible number with which when 60 and 98 are divided it leaves the remainders 3 in each case.

Solution Since 60 and 98 both leave the remainders 3 when divided by such a number.

Therefore $57 = (60 - 3)$ and $95 = (98 - 3)$ will be divisible by the same number without leaving any remainder. That means the HCF of 57 and 95 is 19.

Hence 19 is the highest possible number.

Case 3. Find the largest possible number with which when 38, 66 and 80 are divided the remainder remains the same.

Solution In this case (since we do not know the value of remainder) we take the HCF of the differences of the given numbers.

So the HCF of (66 – 38), (80 – 66), (80 – 38) = HCF of 28, 14, 42 = 14. Hence 14 is the largest possible number which leaves same remainder (= 10) when it divides either 38, 66 or 80.

NOTE In case 1, the remainders are different
In case 2, the remainders are same in each case.
In case 3, the remainders are same in each case, but the value of remainder is unknown.

Least Common Multiple (LCM)

In this chapter I have already discussed the multiples of a number for example integral multiples of 7 are 7, 14, 21, 28, 35, 42, ... and the integral multiples of 8 are 8, 16, 24, 32, 40, 48, 56, ...etc. In short we can write the positive integral multiples of any number N as NK,

where $K = 1, 2, 3, 4, 5, 6, \ldots$

Now if we consider at least two numbers say 2 and 3 then we write the multiples of each as 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30... and 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, ...

Thus it is clear from the above illustrations that some of the multiples of 2 and 3 are common *viz.,* 6, 12, 18, 24, 30...

Again if we consider any 3 numbers say, 3, 5 and 6 then the multiples of each of the 3, 5, 6 are as follows:

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60, ...

5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, ...

6, 12, 18, 24, 30, 36, 42, 48, 54, 60,...

So, we can see that the common multiples of 3, 5 and 6 are 30, 60, 90, 120, ... etc.

But, out of these common multiples, 30 is the least common multiple. Similarly in the previous example, 6 is the least common multiple.

NOTE Generally there is no any greatest common multiple unless otherwise the condition is stated.

Basically, there are two methods to find the LCM.

(i) Factor Method
(ii) Division Method

(i) *Factor Method*

Resolve the given numbers into their prime factors, then take the product of all the prime factors of the first number with those prime factors of second number which are not common to the prime factors of the first number.

Now this resultant product can be multiplied with those prime factors of the third number which are not common to the factors of the previous product and this process can be continued for further numbers if any.

Exp. 1) Find the LCM of 48, 72, 140.

Solution We can write the given numbers as

$$48 = \boxed{2} \times \boxed{2} \times 2 \times 2 \times 3 = \boxed{2} \times \boxed{2} \times 2 \times 3 \times 2$$
$$72 = \boxed{2} \times \boxed{2} \times 2 \times 3 \times 3 = \boxed{2} \times \boxed{2} \times 2 \times 3 \times 3$$
$$140 = \boxed{2} \times \boxed{2} \times 5 \times 7 = \boxed{2} \times \boxed{2} \times 5 \times 7$$

So the LCM of 48, 72 and 140 $= 2 \times 2 \times 2 \times 3 \times 2 \times 3 \times 5 \times 7 = 5040$

Exp. 2) Find the LCM of 42, 63 and 231.

Solution
$$42 = 2 \times 3 \times 7 = \boxed{3} \times \boxed{7} \times 2$$
$$63 = 3 \times 3 \times 7 = \boxed{3} \times \boxed{7} \times 3$$
$$231 = 3 \times 7 \times 11 = \boxed{3} \times \boxed{7} \times 11$$

$\therefore$ The LCM of 42, 63 and 231 $= 3 \times 7 \times 2 \times 3 \times 11 = 1386$

(ii) *Division Method*

First of all write down all the given numbers in a line separated by the comma (,) then divide these numbers by the least common prime factors say 2, 3, 5, 7, 11... to the given numbers, then write the quotients just below the actual numbers separated by comma (,).

If any number is not divisible by such a prime factor then write this number as it is just below itself, then continue this process of division by considering higher prime factors, if the division is complete by lower prime factor, till the quotient in the last line is 1.

Then take the product of all the prime factors by which you have divided the numbers (or quotient) in different lines (or steps). This product will be the LCM of the given numbers.

Exp. 3) Find the LCM of 108, 135 and 162.

Solution

2	108, 135, 162
2	54, 135, 81
3	27, 135, 81
3	9, 45, 27
3	3, 15, 9
3	1, 5, 3
5	1, 5, 1
	1, 1, 1

Thus the required LCM
$= 2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 5$
$= 1620$

Exp. 4) Find the LCM of 420, 9009, 6270.

Solution

2	420, 9009, 6270
2	210, 9009, 3135
3	105, 9009, 3135
3	35, 3003, 1045
5	35, 1001, 1045
7	7, 1001, 209
11	1, 143, 209
13	1, 13 19
19	1, 1, 19
	1, 1, 1

Thus the required LCM $= 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 11 \times 13 \times 19$
$= 3423420$

Exp. 5) Find the least possible number which can be divided by 32, 36 and 40.

Solution The number which is divisible by 32, 36 and 40, it must be the common multiple of all the given numbers. Since we need such a least number then we have to find out just the LCM of 32, 36 and 40.

The LCM of 32, 36 and 40 $= 1440$

Hence, the least possible no. is 1440, which is divisible by all the given numbers.

Exp. 6) What is the least possible number of 5 digits which is divisible by all the numbers 32, 36 and 40.

Solution Since the least possible no. is 1440, but it is a four digit number.

So we can take the integral multiples of 1440 which must be divisible by the given numbers.

Now since the least possible 5 digit number is 10000, so the required number must be equal to or greater than 10,000. So when we multiply 1440 by 7. We get the required result *i.e.,* 10,080. Thus 10080 is the least possible 5 digit number which is divisible by 32, 36 and 40.

Alternatively Divide the least possible 5 digit no.by 1440 and then add the difference of the divisor and remainder to the least possible 5 digit number. This will be the required number.

So,

$$1440 \overline{)10000} (6$$
$$\underline{8640}$$
$$\underline{1360}$$

Now, $1440 - 1360 = 80$

Thus the required number $= 10000 + 80 = 10080$

Exp. 7) Find the largest possible number of 4 digits which is exactly divisible by 32, 36 and 40.

Solution Since the largest 4 digit number is 9999, so the required number can not exceed 9999 any how. Now we take the appropriate multiple of 1440. Since 1440 is divisible by the given numbers, so the multiples of 1440 must be divisible by the given numbers.

Hence $1440 \times 6 = 8640$ is the largest possible number since $1440 \times 7 = 10080$ is greater than 9999 which is not admissible. Hence, 8640 is the largest possible 4 digit no. which is divisible by all the given numbers.

Alternatively Divide the greatest 4 digit number by 1440 (the LCM of the given no.) and then subtract the remainder from the greatest 4 digit number. So,

$$1440 \overline{)9999} (6$$
$$\underline{8640}$$
$$\underline{1359}$$

Hence, the required number $= 9999 - 1359 = 8640$

Exp. 8) Find the number of numbers lying between 1 and 1000 which are divisible by each of 6, 7 and 15.

Solution The least possible number which is divisible by 6, 7, and 15 = LCM of 6, 7, 15 = 210

So, the first such number is 210 and the other numbers are the multiples of 210 *i.e.,* 210, 420, 630, 840.

Thus there are total 4 numbers lying between 1 and 1000 which are divisible by 6, 7 and 15.

Exp. 9) Find the number of numbers lying between 1000 and 1,00,000 which are divisible by 15, 35 and 77 and are even also.

Solution The required numbers must be the multiples of the LCM of 15, 35 and 77. Now the LCM of 15, 35 and 77 = 1155.

So the other numbers are 1155, 2310, 3465, 4620, 5775, ..., 99330

Thus there are total 86 numbers which are divisible by 15, 35 and 77 but only 43 numbers are even *i.e.,* every alternate number is even. Thus there are total 43 required numbers.

Exp. 10) Find the least possible perfect square number which is exactly divisible by 6, 40, 49 and 75.

Solution The required number must be divisible by the given numbers so it can be the LCM or its multiple number.

Now the LCM of 6, 40, 49 and 75 $= 2 \times 2 \times 2 \times 3 \times 5 \times 5 \times 7 \times 7$

But the required number is a perfect square

Thus the LCM must be multiplied by $2 \times 3 = 6$.

Thus the required number

$$= (2 \times 2 \times 2 \times 3 \times 5 \times 5 \times 7 \times 7) \times (2 \times 3)$$
$$= 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 7 = 176400$$

Exp. 11) Three bells in the bhootnath temple toll at the interval of 48, 72 and 108 second individually. If they have tolled all together at 6 : 00 AM then at what time will they toll together after 6 : 00 AM?

Solution The three bells toll together only at the LCM of the times they toll individually.

Thus the LCM of 48, 72 and 108 is 432 seconds.

Therefore all the bells will toll together at 6 : 07 : 12 AM

($\because$ 432 seconds = 7 minuts 12 seconds)

Exp. 12) In the above problem how many times these bells will toll together till the 6 : 00 PM on the same day.

Solution The total time since 6 : 00 AM till 6 : 00 PM

$$= 12 \times 60 \times 60 \text{ seconds.}$$

Now since all these bells toll together at the interval of 432 seconds. So the number of times when they will toll together

$$= \frac{\text{Total time}}{\text{least duration of tolling together}}$$

$$= \frac{12 \times 60 \times 60}{432} = 100$$

but since the bells toll together at 6 : 00 PM also. Hence total 101 (= 100 + 1) times these bells will toll together in the given duration of time.

LCM with Remainders

Case 1. When the remainders are same for all the divisors.

Case 2. When the remainders are different for different divisors, but the respective difference between the divisors and the remainders remains constant.

Case 3. When neither the divisors are same nor the respective differences between divisors and the remainders remain constant.

Case 1

Exp. 13) What is the least possible number which when divided by 24, 32 or 42 in each case it leaves the remainder 5?

Solution Since we know that the LCM of 24, 32 and 42 is divisible by the given numbers. So the required number

$$= (\text{LCM of } 24, 32, 42) + (5) = 672 + 5 = 677$$

Hence such a least possible number is 677.

Exp. 14) In the above question how many numbers are possible between 666 and 8888?

Solution Since the form of such a number is $672m + 5$, where $m = 1, 2, 3, \ldots$

So, the first no. $= 672 \times 1 + 5 = 677$ and the highest possible number in the given range $= 672 \times 13 + 5 = 8736 + 5 = 8741$

Thus the total numbers between 666 and 8888 are 13.

Exp. 15) What is the least possible number which when divided by 21, 25, 27 and 35 it leaves the remainder 2 in each case?

Solution The least possible number

$= (\text{LCM of } 21, 25, 27 \text{ and } 35)\, m + 2$

$= 4725m + 2$

$= 4727$ (for the least possible value we take $m = 1$)

Exp. 16) What is the least possible number which must be added to 4722 so that it becomes divisible by 21, 25, 27 and 35?

Solution The number which is divisible by 21, 25, 27 and 35 is the LCM of 21, 25, 27 and 35 = 4725

So the required number = 4725 − 4722 = 3

Exp. 17) What is the least possible number which when divided by 8, 12 and 16 leaves 3 as the remainder in each case, but when divided by 7 leaves no any remainder?

Solution The possible value $= m\,(\text{LCM of } 8, 12 \text{ and } 16) + 3$

$= m\,(48) + 3$

Now put the value of m such that "$m\,(48) + 3$" becomes divisible by 7. So $3 \times 48 + 3 = 147$ which is the least possible required number.

Case 2

Exp. 18) What is the least possible number which when divided by 18, 35 or 42 it leaves. 2, 19, 26 as the remainders, respectively?

Solution Since the difference between the divisors and the respective remainders is same.

Therefore, the least possible number

$= (\text{LCM of } 18, 35 \text{ and } 42) - 16$

$= 630 - 16$ $[\because (18 - 2) = (35 - 19) = (42 - 26) = 16]$

$= 614$

Exp. 19) What is the least possible number which when divided by 2, 3, 4, 5, 6 it leaves the remainders 1, 2, 3, 4, 5 respectively?

Solution Since the difference is same as

$$(2 - 1) = (3 - 2) = (4 - 3) = (5 - 4)\ (6 - 5) = 1$$

Hence the required number = (LCM of 2, 3, 4, 5, 6) − 1

$= 60 - 1 = 59$

Exp. 20) In the above problem what is the least possible 3 digit number which is divisible by 11?

Solution Since the form of the number is $60m - 1$, where $m = 1, 2, 3, \ldots.$

but the number $(60m - 1)$ should also be divisible by 11 hence at $m = 9$ the number becomes 539 which is also divisible by 11.

Thus the required number = 539.

Exp. 21) Find the least possible 5 digit number which when divided by 2, 4, 6 and 8, it leaves the remainders 1, 3, 5 and 7, respectively.

Solution The possible value = (LCM of 2, 4, 6, 8) $m - 1 = 24m - 1$

Since, the least 5 digit number is 10000. So the required number must be atleast 10000. So putting the value of $m = 417$, we get $(10008 - 1) = 10007$, which is the required number.

Case 3

Exp. 22) What is the least possible number which when divided by 13 it leaves the remainder 3 and when it is divided by 5 it leaves the remainder 2.

Solution Let the required number be N then it can be expressed as follows

$$N = 13k + 3 \quad \ldots(i)$$

and $$N = 5l + 2 \quad \ldots(ii)$$

where k and l are the quotients belong to the set of integers.

Thus $$5l + 2 = 13k + 3 \Rightarrow 5l - 13k = 1 \Rightarrow 5l = 13k + 1$$

$$\Rightarrow \quad l = \frac{13k + 1}{5}$$

Now, we put the value of k such that numerator will be divisible by 5 or l must be integer so considering $k = 1, 2, 3, \ldots$ we find that $k = 3$, l becomes 8. So the number $N = 5 \times 8 + 2 = 42$

Thus the least possible number = 42

To get the higher numbers which satisfy the given conditions in the above problem we just add the multiples of the given divisors (*i.e.*, 13 and 5) to the least possible number (*i.e.*, 42)

NOTE The next higher number = (Multiple of LCM of 13 and 5) + $42 = 65m + 42$

Exp. 23) In the above problem what is the greatest possible number of 4 digits?

Solution Since the general form of this number is $65m + 42$. So by putting $m = 153$ we get $(9945 + 42 =)$ 9987, which is required number.

Hint To get the value of 9945 we can simply divide 9999 (which is the greatest 4 digit number) by 65 and then subtract the remainder from 9999.

Exp. 24) How many numbers lie between 11 and 1111 which when divided by 9 leave a remainder of 6 and when divide by 21 leave a remainder of 12?

Solution Let the possible number be N then it can be expressed as

$$N = 9k + 6 \quad \text{and } N = 21l + 12$$

$$\therefore \quad 9k + 6 = 21l + 12 \Rightarrow 9k - 21l = 6$$

or $$3(3k - 7l) = 6 \text{ or } 3k = 7l + 2 \quad \text{or} \quad k = \frac{7l + 2}{3}$$

So put the min. possible value of l such that the value of k is an integer or in other words numerator (*i.e.*, $7l + 2$) will be divisible by 3.

Thus at $l = 1$, we get $k = 3$ (an integer). So the least possible number $N = 9 \times 3 + 6 = 21 \times 1 + 12 = 33$.

Now the higher possible values can be obtained by adding 33 in the multiples of LCM of 9 and 21. *i.e.*, The general form of the number is $63m + 33$. So the other number in the given range including 33 are 96, 159, 222, 285, 348, ..., 1104. Hence there are total 18 numbers which satisfy the given condition.

Important Formula Product of two numbers = Product of their HCF and their LCM.

Exp. 25) The HCF and LCM of the two numbers is 12 and 600 respectively. If one of the numbers is 24, then the other number will be

(a) 300 (b) 400
(c) 1500 (d) none of these

Solution The answer is (a)

since product of numbers = HCF × LCM

$$\therefore \quad N \times 24 = 12 \times 600 \Rightarrow N = 300$$

Successive Division

If the quotient in a division is further used as a dividend for the next divisor and again the latest obtained divisor is used as a dividend for another divisor and so on, then it is called the ''successive division'' *i.e.*, if we divide 625 by 3, we get 208 as quotient and 1 as a remainder then if 208 is divided by another divisor say 4 then we get 52 as a quotient and '0' (zero) as remainder and again, if we divide 52 by another divisor say 6 we get 8 as quotient and 4 as a remainder *i.e.*, we can represent it as following

$$\begin{array}{r|l}
3 & 625 \\ \hline
4 & 208 \quad \to 1 \text{(Remainder)} \\ \hline
6 & 52 \quad \to 0 \text{(Remainder)} \\ \hline
 & 8 \quad \to 4 \text{(Remainder)}
\end{array}$$

Now you can see that the quotient obtained in the first division behaves as a dividend for another divisor 4. Once again the quotient 52 is treated as a dividend for the next divisor 4. Thus it is clear from the above discussion as

Dividend	Divisor	Quotient	Remainder
625	3	208	1
208	4	52	0
52	6	8	4

So the 625 is successively divided by 3, 4 and 6 and the corresponding remainders are 1, 0 and 4.

Exp. 1) The least possible number of 3 digits when successively divided by 2, 5, 4, 3 gives respective remainders of 1, 1, 3, 1 is :

(a) 372 (b) 275
(c) 273 (d) 193

Solution The problem can be expressed as

$$\begin{array}{r|l}
2 & A \\ \hline
5 & B \quad \to 1 \\ \hline
4 & C \quad \to 1 \\ \hline
3 & D \quad \to 3 \\ \hline
 & E \quad \to 1
\end{array} \left.\vphantom{\begin{array}{l}1\\1\\3\\1\end{array}}\right\} \text{Remainder}$$

So it can be solved as

$$(((((E \times 3) + 1)\ 4 + 3)\ 5 + 1)\ 2 + 1) = A$$

(where A is the required number)

So for the least possible number E

$= 1$(the least positive integer)

then $A = (((((1 \times 3) + 1) \times 4 + 3)\ 5 + 1)\ 2 + 1)$

[Since at E = 0, we get a two digit number]

Alternatively We use the following convention while solving this type of problem.

First we write all the divisors as given below then their respective remainders just below them.

$$\begin{array}{cccccc}
2 & \nwarrow & 5 & \nwarrow & 4 & \nwarrow \quad 3] \text{ Divisors} \\
\downarrow 1 & \searrow & \downarrow 1 & \searrow & \downarrow 3 & \searrow \quad 1] \text{ Remainders}
\end{array}$$

Where the arrow downwards means to add up and the arrow slightly upward (at an angle of 45°) means multiplication.

So we start from the right side remainder and move towards left since while writing down the divisors and remainders we write the first divisor first (*i.e.,* leftmost), second divisor at second position (from the left) and so on.

Now solve it as : **Step 1.** $(1 \times 4) + 3 = 7$

Step 2. $7 \times 5 + 1 = 36$

Step 3. $36 \times 2 + 1 = 73$

or $((((1 \times 4) + 3)\ 5 + 1)\ 2 + 1) = 73$

but we are required to find a three digit number so the next higher numbers can be obtained just by taking the multiples of the product of the divisors and then adding to it the least such number.

The next higher number

$$= (2 \times 5 \times 4 \times 3)m + 73$$
$$= 120m + 73$$

So by putting $m = 1, 2, 3...$ we can get the higher possible numbers.

Now we just need a least possible 3 digit number so we can get it by putting $m = 1$

Hence the required number

$$= 120 \times 1 + 73 = 193$$

Hence (d) is the correct answer.

Exp. 2) A number when successively divided by 2, 3 and 5 it leaves the respective remainders 1, 2 and 3. What will be the remainder if this number is divided by 7?

Solution Write the divisors and remainders as given below

$$\begin{array}{ccccc}
2 & \nwarrow & 3 & \nwarrow & 5 \\
\downarrow 1 & \searrow & \downarrow 2 & \searrow & 3
\end{array}$$

then solve it as follows $(((3 \times 3) + 2)\ 2 + 1) = 23$

or
$$3 \times 3 = 9$$
$$9 + 2 = 11$$
$$11 \times 2 = 22$$
$$22 + 1 = 23$$

So, the least possible number is 23 and the higher numbers can be obtained as $(2 \times 3 \times 5)\ m + 23 = 30m + 23$.

So the higher numbers are 53, 83, 113, 143, 173, 213, 333, ...etc.

But when we divide all these possible numbers by 7 we get the different remainders. So we can not conclude the single figure as a remainder.

Exp. 3) A number when divided successively by 6, 7 and 8, it leaves the respective remainders of 3, 5 and 4. What will be the last remainder when such a least possible number is divided successively by 8, 7 and 6?

Solution First we find the actual least possible no. then we move further

$$\begin{array}{ccccc}
6 & \nwarrow & 7 & \nwarrow & 8 \\
\downarrow 3 & \searrow & \downarrow 5 & \searrow & 4
\end{array}$$

$$((((4 \times 7) + 5) \times 6) + 3) = (((28 + 5) \times 6) + 3)$$
$$= 33 \times 6 + 3 = 198 + 3 = 201$$

Now we divide 201 successively by 8, 7 and 6.

$$\begin{array}{r|l}
8 & 201 \\ \hline
7 & 25 \quad \to 1 \text{ (Remainder)} \\ \hline
6 & 3 \quad \to 4 \text{ (Remainder)} \\ \hline
 & 0 \quad \to 3 \text{ (Remainder)}
\end{array}$$

So 3 is the last remainder.

Exp. 4) How many numbers lie between 100 and 10000 which when successively divided by 7, 11 and 13 leaves the respective remainders as 5, 6 and 7?

Solution The least possible number can be obtained as

$$\begin{array}{ccccc}
7 & \nwarrow & 11 & \nwarrow & 13 \\
\downarrow 5 & \searrow & \downarrow 6 & \searrow & 7
\end{array}$$

$$(((7 \times 11) + 6)\ 7 + 5) = ((77 + 6)\ 7 + 5)$$
$$= (83 \times 7 + 5) = (581 + 5) = 586$$

The general form for the higher numbers is

$$(7 \times 11 \times 13)m + 586 = (1001)m + 586$$

So, the numbers can be obtained by considering $m = 0, 1, 2, 3, ...$ so the first number is 586 and the last number is 9595 which can be attained at $m = 9$. So there are total 10 such numbers lying between 100 and 10000.

Introductory Exercise 1.6

1. HCF of 1007 and 1273 is :
(a) 1 (b) 17 (c) 23 (d) 19

2. The GCD of two whole numbers is 5 and their LCM is 60. If one of the numbers is 20, then other number would be :
(a) 25 (b) 13 (c) 16 (d) 15

3. The number of possible pairs of numbers, whose product is 5400 and HCF is 30 :
(a) 1 (b) 2 (c) 3 (d) 4

4. The number of pairs lying between 40 and 100, such that HCF is 15, is :
(a) 3 (b) 4 (c) 5 (d) 6

5. A merchant has 140 litres, 260 litres and 320 litres of three kinds of oil. He wants to sell the oil by filling the three kinds of oil separately in tins of equal volume. The volume of such a tin is :
(a) 20 litres (b) 13 litres
(c) 16 litres (d) 70 litres

6. If x and y be integer such that $3x + 2y = 1$ then consider the following statements regarding x and y :
1. x and y can be found using euclidean algorithm.
2. x and y are positive.
3. x and y are uniquely determined.

Of these statements :
(a) 1 and 2 are correct (b) 1 alone is correct
(c) 1 and 3 are correct (d) 1, 2 and 3 are correct

7. If d is the HCF of a and b, then $d = \lambda a + \mu b$ where :
(a) λ and μ are uniquely determined
(b) λ and μ are both positive
(c) λ and μ are both negative
(d) one of the λ and μ is negative and the other is positive

8. The product of two numbers is 84 and their HCF is 2. Find the numbers of such pairs.
(a) 8 (b) 5 (c) 4 (d) 2

9. The product of two numbers is 15120 and their HCF is 6, find the number of such pairs.
(a) 5 (b) 6 (c) 3 (d) 8

10. The number of pairs of two numbers whose product is 300 and their HCF is 5 :
(a) 2 (b) 3
(c) 4 (d) can't be determined

11. The largest possible number by which when 76, 132 and 160 are divided the remainders obtained are the same is :
(a) 6 (b) 14
(c) 18 (d) none of these

12. Find the number of pairs of two numbers whose HCF is 5 and their sum is 50.
(a) 4 (b) 3
(c) 2 (d) none of these

13. The largest possible length of a tape which can measure 525 cm, 1050 cm and 1155 cm length of cloths in a minimum number of attempts without measuring the length of a cloth in a fraction of the tape's length
(a) 25 (b) 105
(c) 75 (d) none of these

14. In the above question minimum how many attempts are required to measure whole length of cloths?
(a) 16 (b) 26
(c) 24 (d) 30

15. Minimum how many similar tiles of square shape are required to furnish the floor of a room with the length of 462 cm and breadth of 360 cm?
(a) 4420 (b) 4220
(c) 4120 (d) 4620

16. The ratio of two numbers is 15 : 11. If their HCF be 13 then these numbers will be :
(a) 15 : 11 (b) 75 : 55
(c) 105 : 77 (d) 195 : 143

17. The three numbers are in the ratio 1 : 2 : 3 and their HCF is 12. These numbers are :
(a) 4, 8, 12 (b) 5, 10, 15
(c) 24, 48, 72 (d) 12, 24, 36

18. There are three drums with 1653 litre, 2261 litre and 2527 litre of petrol. The greatest possible size of the measuring vessel with which we can measure up the petrol of any drum, while every the vessel must be completely filled:
(a) 31 (b) 27
(c) 19 (d) 41

19. Two pencils are of 24 cm and 42 cm. If we want to make them of equal size then minimum no. of similar pencils is
(a) 6 (b) 11
(c) 12 (d) none of these

20. The HCF of two numbers is 27 and their sum is 216. These numbers are :
(a) 27, 189 (b) 81, 189
(c) 108, 108 (d) 154, 162

21. Mr. Baagwan wants to plant 36 mango trees, 144 orange trees and 234 apple trees in his garden. If he wants to plant the equal no. of trees in every row, but the rows of mango, orange and apple trees will be separate, then the minimum number of rows in his garden is :
(a) 18 (b) 23
(c) 36 (d) can't be determined

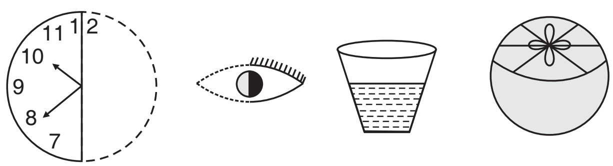
22. If the product of the HCF and the LCM of 3 natural numbers p, q, r equals pqr, then p, q, r must be :
(a) such that $(p, q, r) = 1$
(b) prime number
(c) odd number
(d) such that $(p, q) = (p, r) = (q, r) = 1$

1.7 Fractions And Decimal Fractions

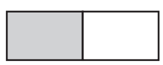
Suppose you have borrowed Rs. 20 from your friend in the last month. Now he has asked you to return his money. But you are paying him only Rs. 10 and the rest amount you want to return in the next month.

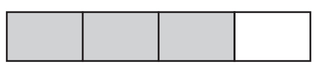
It means you are paying the total amount not in a one lot but in a fraction that means ‘‘in parts’’.

From figures you can see that these things are in fraction.



So, we can say that when any unit of a thing is divided into equal parts and some parts are considered, then it is called a fraction. For example $\frac{1}{2}, \frac{3}{4}, \frac{2}{5}, \frac{3}{7}$ etc.

$\frac{1}{2}$ is read as one half 

$\frac{3}{4}$ is read as three fourth 

Denominator : The lower value indicates the number of parts into which the whole thing or quantity is being equally divided and it is known as Denominator.

Numerator : The upper value indicates the number of parts taken into consideration (or for use) out of the total parts in known as Numerator.

So in the fractions $\frac{2}{3}, \frac{5}{7}, \frac{1}{4}$ etc. 2, 5, 1 are the numerators and 3, 7, 4 are the denominators. The numerators and denominators of a fraction are also called the ‘terms’ of a fraction.

Proper Fraction : A fraction whose numerator is less then its denominator,but not equal to zero) is called a proper fraction. For example $\frac{1}{2}, \frac{3}{4}, \frac{2}{7}, \frac{11}{20}$, etc.

Improper Fraction : A fraction whose numerator is equal to or greater than its denominator is called an improper fraction. For example $\frac{7}{2}, \frac{8}{5}, \frac{215}{15}, \frac{63}{15}$ etc.

NOTE 1. Every natural number can be expressed as a fraction, for example $7 = \frac{7}{1}, 8 = \frac{8}{1}, 11 = \frac{11}{1}$ etc. which are improper fraction.
2. When the numerator and denominator of a fraction are equal, then it is equal to unity.
3. The denominator of a fraction can never be equal to zero.

Mixed Fraction : A number which consists of two parts
(i) a natural number (ii) a proper fraction
is called a mixed fraction.For example $2\frac{5}{13}, 1\frac{7}{8}, 18\frac{9}{17}$ etc. where 2, 1, 18 are natural numbers and $\frac{5}{13}, \frac{7}{8}$ and $\frac{9}{17}$ are proper fractions. and $2\frac{5}{13} = 2 + \frac{5}{13}, 1\frac{7}{8} = 1 + \frac{7}{8}$ and $18\frac{9}{17} = 18 + \frac{9}{17}$

NOTE 1. Every mixed fraction can be written as an improper fraction and every improper fraction can be written as a mixed fraction as $\frac{45}{7} = 6\frac{3}{7}$
2. The numbers for example $-\frac{2}{3}, \frac{-7}{8}, \frac{-9}{5}$ etc are not the fraction numbers, but these are called as fraction like numbers.

Like Fractions : The fractions whose denominators are same are called like fractions, for example $\frac{3}{11}, \frac{4}{11}, \frac{9}{11}, \frac{8}{11}, \frac{17}{11}$ etc.

Equivalent Fractions : The fractions whose values are same *i.e.,* the ratio is same are called equivalent fractions for example, $\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15} = \frac{12}{18} = \frac{20}{30} = \frac{26}{39}$

It implies that :

1. If we multiply the numerator and the denominator by the same non-zero number, the value of the fraction remains unchanged.
2. If we divide the numerator and the denominator by the same non zero number, the value of the fractions remains unchanged.

Cancellation : Division of the numerator and denominator by the same (non-zero) number is called the cancellation when the numerator and denominator of a fraction have no any common factors between them, then it is said the Reduced or Simplest form of fraction or in lowest term. For example $\frac{2}{3}, \frac{3}{4}, \frac{7}{8}$ etc are the reduced fractions in their lowest term.

Reduction of a Fraction to its Lowest Terms or Simplest Form

1. *HCF Method*

Divide the numerator and denominator both by their HCF. *e.g.,* to reduce $\frac{21}{35}$ to its lowest term just divide 21 and 35 by their HCF. So, $\frac{21 \div 7}{35 \div 7} = \frac{3}{5}$

2. ***Prime Factorisation Method***

Just cancel out the common prime factors of both the numerator and denominator. For example to reduce 42 and 140 we cancel out the common prime factors as

$$\frac{42}{140}=\frac{2\times3\times7}{2\times2\times5\times7}=\frac{3}{10}$$

Reduction of the given fractions into like fractions : Obtain the LCM of the denominators of the given fractions and then make all the denominators equal to the LCM obtained, in such a way that the value of every fraction remains unchanged. For example $\frac{6}{35},\frac{10}{21}$.

Since the LCM of the 35 and 21 is 105, so the new fraction will be as $\frac{6\times3}{35\times3},\frac{10\times5}{21\times5}$

$$\Rightarrow \quad \frac{18}{105},\frac{50}{105}$$

Relationship among fractions : If the denominators of all the fractions are same then the fraction with the greater numerator will be greater.

For example $\frac{8}{19}>\frac{7}{19}>\frac{5}{19}>\frac{4}{19}$

NOTE

1. To equalize the denominator of the fractions, take the LCM of the denominators.
2. (A) If $\frac{x}{y}$ be a proper positive fraction and 'a' be a positive integers when $y \neq 0$

   (i) $\frac{x+a}{y+a}>\frac{x}{y}$ (ii) $\frac{x-a}{y-a}<\frac{x}{y}$

   (B) If 'b' being a positive integer such that $a>b$, then

   (iii) $\frac{x}{y}\times\frac{a}{b}>\frac{x}{y}$ (iv) $\frac{x}{y}\times\frac{b}{a}<\frac{x}{y}$
3. If $\frac{x}{y}$ be an improper positive fraction, then

   (v) $\frac{x+a}{y+a}<\frac{x}{y}$ (vi) $\frac{x-a}{y-a}>\frac{x}{y}$

   (vii) $\frac{x}{y}\times\frac{a}{b}>\frac{x}{y}$ (viii) $\frac{x}{y}\times\frac{b}{a}<\frac{x}{y}$
4. For any fraction $\frac{x}{y}$

   (ix) $\frac{x+nx}{y+ny}=\frac{x}{y}$ (x) $\frac{x-nx}{y-ny}=\frac{x}{y}$

   (xi) $\frac{x\pm m}{y\pm n}\neq\frac{x}{y}$ if $\frac{m}{n}\neq\frac{x}{y}$

Practice Exercise

1. State (a) if the fraction is proper, state (b) if the fraction is improper state (c) if the fraction is mixed, state if (d) none of these

   (i) $\frac{3}{7}$ (ii) $\frac{7}{3}$

   (iii) $\frac{35}{12}$ (iv) $4\frac{3}{11}$
2. State (a) if the fractions are equivalent and state (b) if the fractions are like fractions else state (c)

   (i) $\frac{7}{16},\frac{8}{16},\frac{9}{16},\frac{31}{16}$ (ii) $\frac{-5}{-6},\frac{15}{18},\frac{-45}{-54}$

   (iii) $\frac{3}{5},\frac{7}{10},\frac{14}{20}$ (iv) $\frac{9}{25},\frac{4}{16},\frac{13}{29}$
3. State (a) if the fractions are in ascending order state (b) if the fractions are in descending order, else state (c)

   (i) $\frac{3}{5},\frac{7}{9},\frac{13}{8}$ (ii) $\frac{9}{25},\frac{11}{26},\frac{13}{28}$

   (iii) $\frac{17}{16},\frac{35}{52},\frac{65}{75}$ (iv) $\frac{30}{45},\frac{30}{50},\frac{38}{80}$

Answers

1. (i) a (ii) b (iii) b (iv) c
2. (i) b (ii) a (iii) c (iv) c
3. (i) a (ii) a (iii) a (iv) b

Type of Fraction

Reciprocal Fraction : The reciprocal fraction of a number k be $1/k$ *i.e.,* the product of a reciprocal number with the number itself is unity (*i.e.,* 1). Hence the reciprocal of 6 is $\frac{1}{6}$, the reciprocal of $\frac{1}{8}$ is 8, the reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$.

Compound Fraction : The fraction of a fraction is called its compound fraction. For example $\frac{1}{3}$ of $\frac{1}{2}\left(i.e., \frac{1}{3}\times\frac{1}{2}=\frac{1}{6}\right)$ is the compound fraction.

Complex Fraction : If the numerator or denominator or both of a fraction are fraction then it is called as complex fraction. For example $\frac{2/3}{7/8},\frac{7/5}{2},\frac{3/7}{8/9},\frac{2}{3/7}$ etc.

NOTE

$$\frac{a/b}{c/d}=\frac{a}{b}\times\frac{d}{c}$$
$$\frac{a/b}{c}=\frac{a}{b\times c}$$
$$\frac{a}{b/c}=\frac{a\times c}{b}$$

Continued Fraction : A continued fraction consists of the fractional denominators. For example,

(i) $1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5}}}}$ (ii) $2+\cfrac{3}{5+\cfrac{2}{7+\cfrac{6}{5+\cfrac{2}{3}}}}$

NOTE These fractions are solved starting from the bottom towards upside.

Simplification of Fractions

Exp. 1) Solve the following expressions :

(i) $\frac{3}{5}+\frac{2}{7}$ (ii) $\frac{7}{3}+\frac{9}{2}$ (iii) $8\frac{3}{7}+6\frac{1}{5}$ (iv) $\frac{8}{3}-\frac{13}{6}$

(v) $\frac{6}{55}-\frac{7}{88}$ (vi) $9\frac{3}{5}-5\frac{3}{9}$ (vii) $\frac{16}{5}-\frac{3}{8}+\frac{11}{10}$

(viii) $\frac{193}{75}+\frac{37}{81}-\frac{18}{65}$

Solution (i) $\frac{3}{5}+\frac{2}{7}=\frac{3\times7}{5\times7}+\frac{2\times5}{7\times5}=\frac{21+10}{35}=\frac{31}{35}$

(ii) $\frac{7}{3}+\frac{9}{2}=\frac{7\times2}{3\times2}+\frac{9\times3}{2\times3}=\frac{14+27}{6}=\frac{41}{6}=6\frac{5}{6}$

(iii) $8\frac{3}{7}+6\frac{1}{5}=\frac{59}{7}+\frac{31}{5}=\frac{59\times5}{7\times5}+\frac{31\times7}{5\times7}$

$=\frac{295+217}{35}=\frac{512}{35}=14\frac{22}{35}$

(iv) $\frac{8}{3}-\frac{13}{6}=\frac{8\times2}{3\times2}-\frac{13}{6}=\frac{16}{6}-\frac{13}{6}=\frac{3}{6}=\frac{1}{2}$

(v) $\frac{6}{55}-\frac{7}{88}=\frac{6\times8}{55\times8}-\frac{7\times5}{88\times5}=\frac{48}{440}-\frac{35}{440}=\frac{13}{440}$

(vi) $9\frac{3}{5}-5\frac{3}{9}=\frac{48}{5}-\frac{48}{9}=\frac{48\times9}{5\times9}-\frac{48\times5}{9\times5}$

$=\frac{432}{45}-\frac{240}{45}=\frac{192}{45}=\frac{64}{15}=4\frac{4}{15}$

(vii) $\frac{16}{5}-\frac{3}{8}+\frac{11}{10}=\frac{16\times8}{5\times8}-\frac{3\times5}{8\times5}+\frac{11\times4}{10\times4}$

$=\frac{128-15+44}{40}=\frac{157}{40}=3\frac{37}{40}$

(viii) $\frac{193}{75}+\frac{37}{81}+\frac{18}{65}=\frac{193\times27\times13}{75\times27\times13}+\frac{37\times25\times13}{81\times25\times13}$

$+\frac{18\times81\times5}{65\times81\times5}$

$=\frac{67743+12025+7290}{26325}=\frac{87058}{26325}=3\frac{8083}{26325}$

Exp. 2) Simplify the following expressions :

(i) $\frac{3}{7}\times\frac{5}{4}$ (ii) $\frac{3}{8}\times\frac{9}{16}\times\frac{7}{5}$

(iii) $\frac{9}{7}\times\frac{21}{6}\times\frac{5}{8}\times3\frac{5}{6}$ (iv) $\frac{4}{9}$ of $\frac{729}{48}$

(v) $\frac{1}{2}$ of $\left(\frac{16}{7}+\frac{5}{3}\right)$ (vi) $\frac{2}{3}$ of $21\frac{4}{9}-9\frac{4}{11}$

(vii) $\frac{6}{15}\div\frac{3}{2}$ (viii) $\frac{14}{9}\div\frac{3}{7}$

(ix) $\frac{3}{7}\div\frac{2}{7}\times\frac{21}{8}$ (x) $\frac{9}{5}$ of $\frac{37}{8}\times\frac{36}{74}$

(xi) $5\frac{1}{3}\div\frac{4}{15}$ (xii) $36\div4$ of $\frac{1}{2}+\frac{3}{4}\times\frac{3}{2}$

Solution (i) $\frac{3}{7}\times\frac{5}{4}=\frac{15}{28}$

(ii) $\frac{3}{8}\times\frac{9}{16}\times\frac{7}{5}=\frac{189}{640}$

(iii) $\frac{9}{7}\times\frac{21}{6}\times\frac{5}{8}\times3\frac{5}{6}=\frac{9}{7}\times\frac{21}{6}\times\frac{5}{8}\times\frac{23}{6}=\frac{345}{32}=10\frac{25}{32}$

(iv) $\frac{4}{9}$ of $\frac{729}{48}=\frac{4}{9}\times\frac{729}{48}=\frac{1}{9}\times\frac{729}{12}$

$=\frac{1}{1}\times\frac{81}{12}=\frac{1}{1}\times\frac{27}{4}=\frac{27}{4}=6\frac{3}{4}$

(v) $\frac{1}{2}$ of $\left(\frac{16}{7}+\frac{5}{3}\right)=\frac{1}{2}\times\left(\frac{16}{7}+\frac{5}{3}\right)$

$=\frac{1}{2}\times\left(\frac{48}{21}+\frac{35}{21}\right)=\frac{1}{2}\left(\frac{83}{21}\right)=\frac{83}{42}=1\frac{41}{42}$

(vi) $\frac{2}{3}$ of $21\frac{4}{9}-9\frac{4}{11}=\frac{2}{3}\times21\frac{4}{9}-9\frac{4}{11}$

$=\frac{2}{3}\times\frac{193}{9}-\frac{103}{11}=\frac{386}{27}-\frac{103}{11}$

$=\frac{386\times11}{27\times11}-\frac{103\times27}{11\times27}=\frac{4246-2781}{297}=\frac{1465}{297}=4\frac{277}{297}$

(vii) $\frac{6}{15}\div\frac{3}{2}=\frac{6/15}{3/2}=\frac{6}{15}\times\frac{2}{3}=\frac{4}{15}$

(viii) $\frac{14}{9}\div\frac{3}{7}=\frac{14/9}{3/7}=\frac{14}{9}\times\frac{7}{3}=\frac{98}{27}=3\frac{17}{27}$

(ix) $\frac{3}{7}\div\frac{2}{7}\times\frac{21}{8}=\frac{3}{7}\times\frac{7}{2}\times\frac{21}{8}=\frac{63}{16}=3\frac{15}{16}$

(x) $\frac{9}{5}$ of $\frac{37}{8}\times\frac{36}{74}=\frac{9}{5}\times\frac{37}{8}\times\frac{36}{74}=\frac{81}{20}=4\frac{1}{20}$

(xi) $5\frac{1}{3}\div\frac{4}{15}=\frac{16}{3}\div\frac{4}{15}=\frac{16}{3}\times\frac{15}{4}=\frac{20}{1}=20$

(xii) $36\div4$ of $\frac{1}{2}+\frac{3}{4}\times\frac{3}{2}=36\div4\times\frac{1}{2}+\frac{3}{4}\times\frac{3}{2}$

$=36\div2+\frac{9}{8}=18+\frac{9}{8}=\frac{153}{8}=19\frac{1}{8}$

Exp. 3) Simplify the following expressions :

(a) $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6}}}$ (b) $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}$

(c) $2-\cfrac{1}{3+\cfrac{1}{4-\cfrac{1}{5+\cfrac{1}{6-\cfrac{1}{7}}}}}$

Solution (a) $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6}}}=1+\cfrac{1}{1+\cfrac{1}{7/6}}$

$=1+\cfrac{1}{1+\cfrac{6}{7}}=1+\cfrac{1}{13/7}$

$=1+\frac{7}{13}=\frac{20}{13}=1\frac{7}{13}$

(b) $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}$

$=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{3}{2}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{3}}}$

$=1+\cfrac{1}{1+\cfrac{1}{\frac{5}{3}}}=1+\cfrac{1}{1+\cfrac{3}{5}}=1+\cfrac{1}{\frac{8}{5}}$

$=1+\frac{5}{8}=\frac{13}{8}=1\frac{5}{8}$

(c) $2-\cfrac{1}{3+\cfrac{1}{4-\cfrac{1}{5+\cfrac{1}{6-\cfrac{1}{7}}}}}=2-\cfrac{1}{3+\cfrac{1}{4-\cfrac{1}{5+\cfrac{1}{41/7}}}}$

$=2-\cfrac{1}{3+\cfrac{1}{4-\cfrac{1}{5+\cfrac{7}{41}}}}=2-\cfrac{1}{3+\cfrac{1}{4-\cfrac{1}{212/41}}}$

$=2-\cfrac{1}{3+\cfrac{1}{4-\cfrac{41}{212}}}=2-\cfrac{1}{3+\cfrac{1}{\frac{807}{212}}}$

$=2-\cfrac{1}{3+\cfrac{212}{807}}=2-\cfrac{1}{\frac{2633}{807}}=2-\frac{807}{2633}$

$=\frac{4459}{2633}=1\frac{1826}{2633}$

HCF and LCM of Fractions

HCF of fractions : The greatest common fraction is called the HCF of the given fractions.

$$\text{HCF of fractions}=\frac{\text{HCF of Numerator}}{\text{LCM of denominator}}$$

For example, The HCF of $\frac{4}{3},\frac{4}{9},\frac{2}{15},\frac{36}{21}$

$$=\frac{\text{HCF of } 4,\ 4,\ 2,\ 36}{\text{LCM of } 3,\ 9,\ 15,\ 21}=\frac{2}{315}$$

LCM of fractions : The least possible number of fraction which is exactly divisible by all the given fractions is called the LCM of the fractions.

$$\text{LCM of fractions}=\frac{\text{LCM of Numerator}}{\text{HCF of Denominator}}$$

For example The LCM of $\frac{4}{3},\frac{4}{9},\frac{2}{15},\frac{36}{21}$

$$=\frac{\text{LCM of } 4,\ 4,\ 2,\ 36}{\text{HCF of } 3,\ 9,\ 15,\ 21}$$

$$=\frac{36}{3}=12$$

Exp. 1) Find the smallest positive number which is exactly divisible by $\frac{1}{3},\frac{1}{2},\frac{3}{7}$ and $\frac{4}{11}$.

Solution The LCM of the given numbers will be the required number. So, the LCM of $\frac{1}{3},\frac{1}{2},\frac{3}{7},\frac{4}{11}$

$$=\frac{\text{LCM of } 1,\ 1,\ 3,\ 4}{\text{HCF of } 3,\ 2,\ 7,\ 11}=\frac{12}{1}=12$$

Thus 12 is the smallest positive numbers, which is required.

Exp. 2) Four runners started running the race in the same direction around a circular path of 7 km. Their speeds are 4, 3, 9 and 3.5 km/hr individually. If they have started their race at 6 o'clock in the morning, then at what time will they be at the starting point?

Solution Time required by everyone to complete one revolution individually is $\frac{7}{4},\frac{7}{3},\frac{7}{9},\frac{7}{3.5}$ hours.

Therefore everyone must reach at the starting point after the time of the LCM of the individual time period for one revolution.

So the LCM of $\frac{7}{4},\frac{7}{3},\frac{7}{9},\frac{7}{3.5}=\frac{7}{4},\frac{7}{3},\frac{7}{9},\frac{2}{1}$

$$=\frac{\text{LCM of } 7, 7, 7, 2}{\text{HCF of } 4, 3, 9, 1}=\frac{14}{1}=14 \text{ hours.}$$

Hence after 14 hours *i.e.*, at 8 o'clock in the evening of the same day they will meet at the starting point.

Square, square root, cube and cube root of fractions

Square of a fraction $=\dfrac{\text{Square of numerators}}{\text{Square of denominators}}$

Similarly, Cube of a fraction $=\dfrac{\text{Cube of numerator}}{\text{Cube of denominator}}$

NOTE $\sqrt{ab}=\sqrt{a}\,\sqrt{b}$ and $\sqrt{\dfrac{a}{b}}=\dfrac{\sqrt{a}}{\sqrt{b}}$

$(ab)^2=a^2b^2$ and $\left(\dfrac{a}{b}\right)^2=\dfrac{a^2}{b^2}$

Exp. 3) Evaluate the following :

(a) $\left(\dfrac{3}{4}\right)^2$ (b) $\left(\dfrac{17}{25}\right)^2$ (c) $\left(\dfrac{19}{-31}\right)^2$ (d) $\left(\dfrac{78}{36}\right)^2$

(e) $\left(\dfrac{9}{7}\right)^3$ (f) $\left(-\dfrac{13}{12}\right)^3$ (g) $\left(\dfrac{20}{7}\right)^3$ (h) $\left(6\dfrac{1}{2}\right)^3$

Solution (a) $\dfrac{9}{16}$ (b) $\dfrac{289}{625}$ (c) $\dfrac{361}{961}$ (d) $\dfrac{1521}{324}$

(e) $\dfrac{729}{343}$ (f) $-\dfrac{2197}{1728}$ (g) $\dfrac{8000}{343}$ (h) $\dfrac{2197}{8}$

Exp. 4) Evaluate the following :

(a) $\sqrt{\dfrac{961}{625}}$ (b) $\dfrac{\sqrt{7744}}{20}$ (c) $\dfrac{31}{\sqrt{1681}}$ (d) $\sqrt{\dfrac{1156}{529}}$

(e) $\sqrt{50}\times\sqrt{18}$ (f) $\sqrt{39}\times\sqrt{\dfrac{75}{2197}}$

Solution (a) $\sqrt{\dfrac{961}{625}}=\dfrac{31}{25}$ (b) $\dfrac{\sqrt{7744}}{20}=\dfrac{88}{20}=4\dfrac{8}{20}=4\dfrac{2}{5}$

(c) $\dfrac{31}{\sqrt{1681}}=\dfrac{31}{41}$ (d) $\dfrac{\sqrt{1156}}{\sqrt{529}}=\dfrac{34}{23}$

(e) $\sqrt{50}\times\sqrt{18}=\sqrt{50\times18}=\sqrt{900}=30$

(f) $\sqrt{39}\times\sqrt{\dfrac{75}{2197}}=\sqrt{\dfrac{39\times75}{2197}}=\sqrt{\dfrac{13\times3\times3\times25}{13\times13\times13}}=\dfrac{15}{13}=1\dfrac{2}{13}$

Exp. 5) The value of $\left\{\dfrac{1}{(\sqrt{2}-2)}+\dfrac{1}{(\sqrt{2}+2)}+(\sqrt{2}+2)\right\}$ is :

(a) 2 (b) 4 (c) $2\sqrt{2}$ (d) none

Solution $\left\{\dfrac{(\sqrt{2}+2)+(\sqrt{2}-2)}{(\sqrt{2}-2)(\sqrt{2}+2)}+(\sqrt{2}+2)\right\}=2$

Exp. 6) The value of $\left\{\dfrac{1}{(\sqrt{6}-\sqrt{5})}-\dfrac{1}{(\sqrt{5}-\sqrt{4})}+\dfrac{1}{(\sqrt{4}-\sqrt{3})}-\dfrac{1}{(\sqrt{3}-\sqrt{2})}+\dfrac{1}{(\sqrt{2}-1)}\right\}$ is

(a) 5 (b) 7 (c) $\sqrt{12}$ (d) none

Solution $\dfrac{1}{(\sqrt{6}-\sqrt{5})}-\dfrac{1}{(\sqrt{5}-\sqrt{4})}+\dfrac{1}{(\sqrt{4}-\sqrt{3})}-\dfrac{1}{(\sqrt{3}-\sqrt{2})}+\dfrac{1}{(\sqrt{2}-1)}$

$=(\sqrt{6}+\sqrt{5})-(\sqrt{5}+\sqrt{4})+(\sqrt{4}+\sqrt{3})-(\sqrt{3}+\sqrt{2})+(\sqrt{2}+1)$

$=\sqrt{6}+1$

$\left[\because \dfrac{1}{\sqrt{a}-\sqrt{b}}=\dfrac{(\sqrt{a}+\sqrt{b})}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}=\dfrac{\sqrt{a}+\sqrt{b}}{(a-b)}\right]$

Hence (d) is the correct option.

Practice Exercise

1. Find the value of $\dfrac{1}{\sqrt{9}-\sqrt{4}}$.
2. Find the value of x if $\sqrt{1+\dfrac{x}{169}}=\dfrac{14}{13}$.
3. Find the value of x if $140\sqrt{x}+315=1015$.
4. If $\sqrt{\left(1+\dfrac{27}{169}\right)}=\left(1+\dfrac{x}{13}\right)$ then find the value of x.
5. Find the value of $\left[6\dfrac{1}{4}\div\left\{2\dfrac{1}{4}-\dfrac{1}{2}\left(2\dfrac{1}{2}-\overline{\dfrac{1}{4}-\dfrac{1}{6}}\right)\right\}\right]$.
6. Find the value $\dfrac{3}{2}$ of $\left(\dfrac{4}{3}\div\dfrac{5}{7}\right)+\dfrac{1}{2}\div\left[3\dfrac{4}{5}-\left\{\dfrac{2}{5}\div\left(\dfrac{1}{3}+\dfrac{1}{2}+\overline{\dfrac{1}{5}-\dfrac{1}{6}}\right)\right\}\right]$.
7. Find the value of $1+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{8+\cfrac{1}{16}}}}$.
8. Find the value of $1-\cfrac{1}{2+\cfrac{1}{2-\cfrac{1}{2+\cfrac{1}{2}}}}$.
9. Calculate the value of $\dfrac{2}{5}+2\dfrac{4}{9}\div\left[\left(7\dfrac{5}{12}-5\dfrac{3}{4}\right)\div22\dfrac{1}{2}+10\times\dfrac{5}{18}\right]-\dfrac{4}{5}$.
10. Calculate the value of $\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2-\cfrac{1}{2}}}}$.
11. Calculate the value of $\left(1\dfrac{29}{36}+1\dfrac{7}{11}\times4\dfrac{1}{8}\right)\div\left(5\dfrac{1}{9}-\dfrac{63}{8}\div9\dfrac{9}{20}\right)$.
12. Calculate the value of $\dfrac{4}{2\frac{1}{3}}+\dfrac{3}{1\frac{3}{4}}-\dfrac{5}{3\frac{1}{2}}$.

Answers

1. 1	**2.** 27	**3.** 25	**4.** 1	**5.** 6
6. $2\dfrac{2061}{2170}$	**7.** $1\dfrac{532}{1193}$	**8.** $\dfrac{13}{21}$	**9.** $\dfrac{16}{35}$	**10.** $\dfrac{8}{19}$
11. 2	**12.** 2			

Exp. 1) From a rope of length $38\frac{3}{5}m$, a piece of length $5\frac{3}{38}m$ is cut off. The length of the remaining rope is :

(a) $190\frac{99}{33}$ (b) $33\frac{90}{190}$

(c) $33\frac{99}{190}$ (d) none of these

Solution Total length of the rope = $38\frac{3}{5} = \frac{193}{5}$

The length of the rope which has been removed = $5\frac{3}{38} = \frac{193}{38}$

$\therefore$The length of the remaining part of the rope

$$= \frac{193}{5} - \frac{193}{38} = 33\frac{99}{199}$$

Hence (c) is the correct option.

Exp. 2) The cost of 1 metre cloth is ₹ $21\frac{1}{2}$, then the cost of $\frac{42}{43}$ metre

(a) ₹ 21 (b) ₹ 42

(c) ₹ $\frac{43}{2}$ (d) none of these

Solution $\because$ The price of 1m cloth = $21\frac{1}{2} = \frac{43}{2}$

$\therefore$ The price of $\frac{42}{43}$ m cloth = $\frac{42}{43} \times \frac{43}{2} = 21$

Hence option (a) is correct.

Exp. 3) A drum of kerosene oil is $\frac{3}{4}$ full. When 15 litres of oil is drawn from it, it is $\frac{7}{12}$ full. The capacity of the drum is :

(a) 45 (b) 90

(c) 60 (d) can't be determined

Solution Let the capacity of drum be x

Then $$\frac{3}{4}x - 15 = \frac{7}{12}x \Rightarrow \frac{9x-7x}{12} = 15 \Rightarrow x = 90$$

Thus (b) is the correct option.

Alternatively If you consider option (b) then

$$90 \times \frac{3}{4} - 15 = 90 \times \frac{7}{12}$$

Alternatively The decrease in amount = $\frac{x}{6} = 15 \Rightarrow x = 90$ litre.

Exp. 4) A sum of ₹ 11200 is shared among Mr. Khare, Mr. Patel and Mr. Verma. Mr. Khare gets $\frac{1}{4}$th of it while Mr. Patel gets $\frac{1}{5}$th of it. The amount of Mr. Verma is :

(a) 6006 (b) 6160

(c) 3080 (d) data insufficient

Solution The share of Mr. Khare = $\frac{1}{4}$

The share of Mr. Patel = $\frac{1}{5}$

$\therefore$ The share of Mr. Verma = $1 - \left(\frac{1}{4} + \frac{1}{5}\right) = \frac{11}{20}$

So, the amount of Mr. Verma = $\frac{11}{20} \times 11200 = 6160$

Thus option (b) is correct.

Exp. 5) When Sarvesh travelled 33 km, he found that $\frac{2}{3}$rd of the entire journey was still left. The length of the total journey is :

(a) 66 (b) 132

(c) 99 (d) 100

Solution Since $\frac{2}{3}$rd journey is left, it means Sarvesh has travelled only $\frac{1}{3}$rd of his journey which is equal to 33. Thus *i.e.,*

$$\frac{1}{3}x = 33 \Rightarrow x = 99$$

Hence (b) is the right choice.

Exp. 6) Mrs. Verma earns Rs. 18000 per month. She spends $\frac{7}{12}$ on house hold items and $\frac{1}{8}$ on rest of the things.The amount she saves is :

(a) 7120 (b) 5250

(c) 5520 (d) none of these

Solution Her total expenditure = $\frac{7}{12} + \frac{1}{8} = \frac{17}{24}$

$\therefore$ Her savings = $1 - \frac{17}{24} = \frac{7}{24}$

$\therefore$ Her saving in ₹ = $\frac{7}{24} \times 18000 = 5250$

Hence (b) is the correct option.

Exp. 7) Neha, a working lady, earns ₹ x per month. If she spends $\frac{2}{5}$th of her earning for personal uses and $\frac{3}{4}$th of the personal uses, she spends in entertainment while $\frac{7}{20}$th of the expenditure in entertainment she spends in movies only. Her salary would be , if her expenditure in movies is in integers.

(a) 4225 (b) 2175 (c) 200 (d) 3465

Solution Since her salary is ₹ x then her expenditure in movies is

$$\frac{7}{20} \text{ of } \frac{3}{4} \text{ of } \frac{2}{5} \text{ of } x = \frac{7}{20} \times \frac{3}{4} \times \frac{2}{5} \times x = \frac{21x}{200}$$

Now, In order to $\frac{21x}{200}$ be an integer so x must be equal to 200.

Thus (c) is the correct option.

Decimal Fractions

In the previous topics I have discussed the integral and fractional numbers.

Now in the ongoing discussion, I am extenting the theory of numbers as ''decimal numbers''.

In the number 8637, the place value of 7 is $7 \times 1 = 7$, the place value of 3 is $10 \times 3 = 30$, the place value of 6 is $6 \times 100 = 600$ and the place vale of 8 is $8 \times 1000 = 8000$.

Thus, as we move from right to left each next digit is multiplied by 10, 100, 1000, 10000, 100000 etc.

It means moving from left to right the place value of each digit is divided by 10, 100, 1000, etc.....

In the above number 7 is known as unit digit (the right most digit), 3 is known as tens digit, 6 as hundreds digit and 8 as thousands digit.

Now if we move towards the right of unit place, the place value of the digit next to the right of unit digit (is the tenths digit) is being divided by 10, while the place value of the next digit is the hundredth digit is being divided by 100 and so on.

So, for our convenience we put a small dot just right to the unit digit and it is known as decimal point.

Hence, the place value of a digit in a number depends on the place or position with respect to the unit digit position while the **face value** of a digit is always fixed as the face value of 1 is 1, 2 is 2, 3 is 3, 4 is 4 and so on.

Place Value Chart

Places	Ten thousands	Thousands	Hundreds	Tens	Unit	Decimal point	Tenths	Hundredths	Thousandths
Value	10000	1000	100	10	1	.	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$

Thus a decimal number say 4637.938 can be expressed as

$$4 \times 1000 + 6 \times 100 + 3 \times 10 + 7 \times 1 + \frac{9}{10} + \frac{3}{100} + \frac{8}{1000}.$$

NOTE The digits written in the right side of decimal point read separately as we read 2.359 as two point three five nine.

Thus the fraction whose denominators are 10, 100, 1000 etc. are called decimal fractions.

The number left to the decimal point is a whole or integral number, while the number to right to the decimal point is a **decimal number.** In the number 735.82, 735 is a whole number and .82 is the **decimal number**.

Conversion of Decimal Numbers into Decimal Fractions

Write the number of zeros as the number of digits in the decimal number preceded by 1, below the actual number without decimal point as

$$23.6 = \frac{236}{10}, \qquad 4.579 = \frac{4579}{1000}$$

$$75.89 = \frac{7589}{100}, \quad 0.18870 = \frac{18870}{100000} = \frac{1887}{10000}$$

Conversion of Decimal Fractions into Decimal Numbers

Count the number of zeros in the denominator and then count the same number of digits in the numerator starting from the unit digit of the numerator moving to the left and then place the decimal point as :

$$\frac{2375}{10} = 237.5, \quad \frac{2375}{100} = 23.75, \quad \frac{2375}{1000} = 2.375,$$

$$\frac{2375}{100000} = 0.02375, \quad \frac{2375}{10000000} = 0.0002375 \text{ etc.}$$

NOTE $0.35 = 0.350 = 0.3500 = 0.35000$ etc. because increasing the number of zeros in the rightmost of a decimal number is inconsiderable.

For example $0.35 = \frac{35}{100}$; $0.350 = \frac{350}{1000} = \frac{35}{100}$

and $0.3500 = \frac{3500}{10000} = \frac{35}{100}$

Some Mathematical Operations on Decimal Numbers

(i) **Addition and Subtraction :** The decimal numbers are written in such a manner that decimal points of all the numbers fall in one column or fall below one another. For example

$$\begin{array}{r} 2.358 \\ 709.21 \\ 0.35 \\ +\ 2.0067 \\ \hline 713.9247 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{r} 635.888 \\ -\ 28.0125 \\ \hline 607.8755 \\ \hline \end{array}$$

(ii) **Multiplication :** Suppose there is no decimal point as natural numbers and then multiply them. Then put the decimal point in the product as the sum of the decimal places in the multiplicands. For example

		Decimal Places
(i)	$23 \times 1.1 = 25.3$	$0 + 1 = 1$
(ii)	$3.8 \times 4.6 = 17.48$	$1 + 1 = 2$
(iii)	$2.456 \times 7.8 = 19.1568$	$3 + 1 = 4$
(iv)	$8.125 \times 100 = 812.500$	$(3 + 0) = 3$
(v)	$75 \times 0.3 = 22.5$	$(0 + 1) = 1$

(iii) **Division :** Make the divisor as a natural number by shifting the decimal in the right hand side equally in dividend and divisor. Now divide the resultant dividend by this resultant divisor as usual.

For Example :

$$65 \div 1.3 = \frac{65}{1.3} = \frac{650}{13} = 50$$

$$235 \div 4.7 = \frac{235}{4.7} = \frac{2350}{47} = 50$$

$$1.17 \div 13 = \frac{1 \cdot 17}{13} = 0.09$$

$$\left[\because \frac{1 \cdot 17}{13} = \frac{117}{13} \times \frac{1}{100} = 9 \times \frac{1}{100}\right]$$

$$172.8 \div 0.12 = \frac{172.8}{0.12} = \frac{17280}{12} = 1440$$

$$1 \cdot 068 \div 8.9 = \frac{1.068}{8.9} = \frac{10.68}{89} = 0.12$$

$$78 \div 0.0039 = \frac{78}{0.0039} = \frac{780000}{0039} = \frac{780000}{39} = 20000$$

Recurring Decimal A decimal number in which a digit or a set of digits repeats regularly, over a constant period, is called a recurring decimal or periodic decimal. For example

2.33333..., 7.5555, ..., 108.232323...

9.142857142857142857.....

where $2.3333..... = 2.\overline{3}$ or $2.\dot{3}$

$7.5555... = 7.\overline{5}$ or $7.\dot{5}$

$108.232323 ... = 108.\overline{23}$

and $9.142857142857142857\ldots$

$$= 9.\overline{142857} = 9.\dot{1}4285\dot{7}$$

So the recurring of the decimal number is expressed by putting the bar or dot over the period of a recurring decimal.

Pure Recurring decimal

A decimal fraction in which all the figures (or digits) occur repeatedly, is called a pure recurring decimal as 7.4444, ..., 2.666, ..., 9.454545, ... etc.

Mixed Recurring decimal : A decimal, number in which some of the digits do not recur is called a mixed recurring decimal for example 327.63454545...

Non-recurring decimals : A decimal number in which there is no any regular pattern of repetition of digits after decimal point is called a non-recurring decimal. *e.g.,* 3.2466267628....

Conversion of Recurring Decimal into Vulgar Fraction

(a) *Pure Recurring Decimal*

Write down as many 9's in the denominator as the number of digits in the period of decimal number, below the given decimal number, for example,

(i) $0.77777... = 0.\overline{7} = \frac{7}{9}$

(ii) $0.545454... = 0.\overline{54} = \frac{54}{99}$

(iii) $2.357357357357... = 2.\overline{357} = 2 + 0.\overline{357}$

$$= 2 + \frac{357}{999} = \frac{2355}{999}$$

(iv) $0.768397683976839... = 0.\overline{76839} = \frac{76839}{99999}$

(b) *Mixed Recurring Decimal*

Step 1. Write down the number and then subtract from it the number formed by the non recurring digits after decimal point.

Step 2. Write down as many 9s as there are digits in the period of the recurring decimal followed by as many zeros as there is number of digits which are not recurring after the decimal point.

Step 3. Divide the value of step 1 by the value obtained in step 2.

For example :

1. $0.17777... = 0.1\overline{7} = \frac{17-1}{90} = \frac{16}{90} = \frac{8}{45}$

2. $0.8737373... = 0.8\overline{73} = \frac{873-8}{990} = \frac{865}{990} = \frac{173}{198}$

3. $0.78943943943... = 0.78\overline{943} = \frac{78943-78}{99900}$

$$= \frac{78865}{99900} = \frac{15773}{19980}$$

4. $0.250121212\ldots = 0.250\overline{12} = \frac{25012-250}{99000}$

$$= \frac{24762}{99000} = \frac{12381}{49500}$$

5. $5.00\overline{72} = 5 + 0.00\overline{72} = 5 + \left(\frac{72-00}{9900}\right)$

$$= 5 + \frac{72}{9900} = 5 + \frac{2}{275} = 5\frac{2}{275}$$

Addition and Subtraction of Recurring Decimals

Exp. 1) $5.7\overline{32} + 8.\overline{613}$

Solution $5.7\overline{32} + 8.\overline{613} = 5 + 0.7\overline{32} + 8 + 0.\overline{613}$

$$= 5 + \frac{732-7}{990} + 8 + \frac{613}{999} = 5 + \frac{725}{990} + 8 + \frac{613}{999}$$

$$= 13 + \frac{725}{990} + \frac{613}{999} = 13 + \frac{145}{198} + \frac{613}{999}$$

$$= 13 + \frac{145 \times 3 \times 37 + 613 \times 2 \times 11}{2 \times 3 \times 9 \times 11 \times 37}$$

$$= \frac{13 \times 2 \times 3 \times 9 \times 11 \times 37 + 145 \times 3 \times 37 + 613 \times 2 \times 11}{2 \times 3 \times 9 \times 11 \times 37}$$

$$= \frac{285714 + 16095 + 13486}{21978} = \frac{315295}{21978} = 14.3459368$$

Alternatively

Step 1. Express the numbers without bar as

5.732323232 + 8.613613613613

Step 2. Write the numbers as one above other *i.e.,*

5.732323232
8.613613613

Step 3. Divide this number into two parts. In the first part *i.e.,* left side write as many digits as there will be integral value with non recurring decimal. In the right side write as many digits as the LCM of the number of recurring digits in the given decimal number *e.g.,*

5.7	323232	(Since 5.7 is the integral + non-recurring part]
8.6	136136	(The LCM of 2 and 3 is 6]

Step 4. Now add or subtract as usual.

5.7	323232
8.6	136136
14.3	459368

Step 5. Put the bar over the digits which are on the right side in the resultant value.

$14.3\overline{459368}$

Thus $5.7\overline{32} + 8.\overline{613} = 14.3\overline{459368}$

Exp. 2) Solve the following : $19.3\overline{68421} - 16.20\overline{53}$

Solution 19.3684216842168421 – 16.2053535353...

= 19.3684216842168421

= 16.205353535353535353

19.36	8421684216
– 16.20	5353535353
3.16	3068148863

(Since 16.20 is integral part with non recurring digits]

[The LCM of 5 and 2 is 10]

In the first number there are 5 recurring digits and in the second number there are 2 recurring digits]

$$19.3\overline{68421} - 16.20\overline{53} = 3.16\overline{3068148863}$$

Multiplication and Division of Recurring Decimals

It can be done as usual. Just convert the decimals into vulgar fractions and then operate as required.

Exp. 1) $97.2\overline{81} \times 100 = 97.2818181 \times 100$

$$= 9728.181818 = 9728.1\overline{81}$$

(Remember that the set of recurring digits is not altered as in the above problem the set of recurring digits will remain $\overline{81}$ but not $\overline{18}$ since initially it was $\overline{81}$.)

Exp. 2) $25.6\overline{32} \times 55 = (25 + 0.6\overline{32}) \times 55$

$$= 25\frac{632-6}{990} \times 55 = 25\frac{626}{990} \times 55$$

$$= 25\frac{313}{495} \times 55$$

$$= \frac{12688}{495} \times 55$$

$$= \frac{12688}{9} = 1409.7777$$

$$= 1409.\overline{7}$$

Exp. 3) $13.00\overline{5} \times 20 \times 8.4\overline{23} = 13\frac{5}{900} \times 20 \times 8\frac{423-4}{990}$

$$= 13\frac{1}{180} \times 20 \times 8\frac{419}{990}$$

$$= \frac{2341}{180} \times 20 \times \frac{8339}{990} = \frac{2341 \times 8339}{9 \times 990}$$

$$= \frac{19521599}{9 \times 990} = 2190.9763187$$

Exp. 4) $0.08\overline{9} \div 100 = 0.089999\ldots \div 100$

$$= 0.00089999\ldots = 0.0008\overline{9}$$

Exp. 5) $53.08\overline{53} \div 6 = \dfrac{53 + 0.08\overline{53}}{6} = \dfrac{53 + \frac{0853-08}{9900}}{6}$

$$= \frac{53 + \frac{845}{9900}}{6} = \frac{53 + \frac{169}{1980}}{6}$$

$$= \frac{\left(\frac{105109}{1980}\right)}{6} = \frac{105109}{1980 \times 6} = \frac{105109}{11880}$$

$$= 8.847558922558922558922$$

$$= 8.847\overline{558922}$$

Alternatively $53.08\overline{53} \div 6 = 53.08535353... \div 6$

$6 \overline{)53.08535353...}(8.847558922558922$

48
50
48
28
24
45
42
33
30
35
30
53
48
55
54
13
12
15
12
33
30
35
30
53
48
55
54
1 etc.

Square of Decimals

First we assume that there is no decimal point and then square up the given decimal number. But at last we put the dot (as decimal point), counting the digits from the rightmost digit. The no. of places will be double as that of the given number. **For example :**

1. $(2.3)^2 = 5.29$ $[(23)^2 = 529]$

2. $(1.07)^2 = 1.1449$ $[(107)^2 = 11449]$

3. $(0.11)^2 = 0.0121$ $[(11)^2 = 121]$

4. $(1.352)^2 = 1.827904$ $[\because (1352)^2 = 1827904]$

5. $0.1 = 0.01$ $[\because (1)^2 = 1]$

6. $(9.99)^2 = 99.8001$ $[(999)^2 = 998001]$

Square root of Decimals

The process of finding the square root is same as that of integers. Here we put the decimal point as in the division of decimal numbers.

For example :

1. Find the square root of 5.29

	2.3
2	5.29
2	4
43	129
3	129
	×

$\therefore \sqrt{5.29} = 2.3$

2. Find the square root of 0.0121

	0.11
0	0.0121
0	0
01	01
1	01
21	21
1	21
	×

$\therefore \sqrt{0.0121} = 0.11$

3. Find the square root of 536.85374

	23.1701
2	536.85374
2	4
43	136
3	129
461	785
1	461
4627	32437
7	32389
463401	484000
01	463401

The process can be continued if required. But generally we stop our calculation after 2-3 places of decimal point.

NOTE To find the nth power of a decimal number in which the decimal point is placed at m places before the rightmost digit, we simply solve it considering as an integer then we put the decimal point in the resultant value before the $m.n$ places from the right most digit, where m, n are positive integers.

For example : $(2.13)^4 = 20.58346161$

[The number of places of decimal point $= 2 \times 4 = 8$]

HCF and LCM of Decimals

HCF

Step 1. First of all equate the number of places in all the numbers by using zeros, wherever required.

Step 2. Then considering these numbers as integers find the HCF of these numbers.

Step 3. Put the decimal point in the resultant value as many places before the right most digit as that of in the every equated number.

Exp. 1) Find the HCF of 0.0005, 0.005, 0.15, 0.175, 0.5 and 3.5.

Solution

0.0005 ⇒ 5
0.0050 ⇒ 50
0.1500 ⇒ 1500
0.1750 ⇒ 1750
0.5000 ⇒ 5000
3.5000 ⇒ 35000

Then the HCF of 5, 50, 1500, 1750, 5000 and 35000 is 5.
So the HCF of the given numbers is 0.0005 (since there are four digits in all the adjusted (or equated) decimal places.

Exp. 2) Find the HCF of 0.9, 0.36 and 1.08.

Solution
$$\left.\begin{array}{r}0.9\\0.36\\1.08\end{array}\right\}\rightarrow\left.\begin{array}{r}0.90\\0.36\\1.08\end{array}\right\}\rightarrow\begin{array}{r}90\\36\\108\end{array}$$

Now the HCF of 90, 36 and 108 is 18. So the required HCF of the given numbers is 0.18. (Since there are two digits after decimal places in every number in second step)

LCM

Step 1. First of all equate the no. of places in all the given numbers by putting the minimum possible number of zeros at the end of the decimal numbers, wherever required.

Step 2. Now consider the equated numbers as integers and then find the LCM of these numbers.

Step 3. Put the decimal point in the LCM of the numbers as many places as that of in the equated numbers.

Exp. 1) Find the LCM of 1.8, 0.54 and 7.2

Solution
$$\left.\begin{array}{r}1.8\\0.54\\7.2\end{array}\right\}\rightarrow\left.\begin{array}{r}1.80\\0.54\\7.20\end{array}\right\}\rightarrow\begin{array}{r}180\\54\\720\end{array}$$

Now the LCM of 180, 54 and 720 is 2160
Therefore the required LCM is 21.60.

Exp. 2) Find the LCM of 4.44, 37 and 55.5

Solution
$$\left.\begin{array}{r}4.44\\37.0\\55.5\end{array}\right\}\rightarrow\left.\begin{array}{r}4.44\\37.00\\55.50\end{array}\right\}\rightarrow\begin{array}{r}444\\3700\\5550\end{array}$$

Now the LCM of 444, 3700, 5550 is 11100.
Hence the required LCM is 111.00 = 111.

Introductory Exercise 1.7

1. $\frac{-2}{3}, \frac{5}{2}, \frac{-1}{6}$ in descending order can be arranged as :
(a) $\frac{5}{2}, \frac{-1}{6}, \frac{-2}{3}$ (b) $\frac{-2}{3}, \frac{5}{2}, \frac{-1}{6}$
(c) $\frac{-1}{6}, \frac{5}{2}, \frac{-2}{3}$ (d) $\frac{5}{2}, \frac{-2}{3}, \frac{-1}{6}$

2. Which one of the following represents the numbers $-\frac{3}{7}, \frac{2}{3}$ and $\frac{-1}{3}$ in descending order?
(a) $\frac{-1}{3}, \frac{-3}{7}, \frac{2}{3}$ (b) $\frac{2}{3}, \frac{-1}{3}, \frac{-3}{7}$
(c) $\frac{-3}{7}, \frac{2}{3}, \frac{-1}{3}$ (d) $\frac{2}{3}, \frac{-3}{7}, \frac{-1}{3}$

3. Find the greatest among 1029/1025, 1030/1026, 256/255, 1023/1019
(a) 1029/1025
(b) 1030/1026
(c) 256/255
(d) 1023/1019

4. The fundamental arithmetical operations on 2 recurring decimals can be performed directly without converting them to vulgar fractions :
(a) only in addition and subtraction
(b) only in addition and multiplication
(c) only in addition, subtraction and multiplication
(d) in all the four arithmetical operations

5. Find the value of x and y in the given equation $5\frac{7}{x} \times y\frac{1}{13} = 12$:
(a) –2, 9 (b) 9, 2
(c) 4, 3 (d) 9, 4

6. The square root of $32\frac{137}{484}$ is :
(a) $5\frac{15}{22}$
(b) $15\frac{5}{32}$
(c) $5\frac{15}{28}$
(d) none of the above

7. $1 \div \cfrac{1}{1 \div \cfrac{1}{1 \div \cfrac{1}{3}}}$ is equal to :
(a) $\frac{1}{3}$ (b) 1
(c) 3 (d) $1\frac{1}{3}$

8. The value of $\dfrac{4\frac{1}{7} - 2\frac{1}{4}}{3\frac{1}{2} + 1\frac{1}{7}} \div \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{5 - \cfrac{1}{5}}}}$:
(a) 7/29 (b) 5/6
(c) 1 (d) 9/13

9. Mallika travelled $\frac{5}{16}$ of his journey by coach and $\frac{7}{20}$ by rail and then she walked the remaining 10 km. How far did she go altogether?
(a) $27\frac{17}{29}$ (b) $33\frac{7}{27}$
(c) $19\frac{7}{27}$ (d) $29\frac{17}{27}$

10. For every positive integer x_n find the value of $\frac{x_1}{x_2}$

if $x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{x_4 + \dots}}} = \sqrt[3]{10}$

(a) 1/3 (b) 3/10
(c) 2/5 (d) none of these

11. Vishvamitra has some guavas of same shape, size and colour. He wants to distribute them equally among some of his chosen disciples. In order to do this he can cut a guava into uniform pieces only. Which of the following statements is/are false?

(i) He can divide 7 guavas equally among his 12 disciples, such that no any guava is cut into more than 6 pieces.

(ii) He can divide 6 guavas equally among his 12 disciples, such that no any guava is cut into more than 3 pieces.

(iii) He can divide 11 guavas equally among his 15 disciples, such that no any guava is cut into more than 6 pieces.

(iv) He can divide 7 guavas equally among his 15 disciples, such that no any guava is cut into more than 6 pieces.

(a) (i) (b) (ii)
(c) (iii) (d) (iv)

12. The expression $33.33 \div 1.1$ simplifies as :

(a) 33.3 (b) 303
(c) 33.0 (d) 30.3

13. Which one of the following is not correct?

(a) $\sqrt{0.4096} = 0.64$ (b) $\sqrt{40.96} = 6.4$
(c) $\sqrt{0.04096} = 0.064$ (d) $\sqrt{4096} = 64$

14. The value of $\sqrt{900} + \sqrt{0.09} - \sqrt{0.000009}$ is :

(a) 30.297 (b) 30.197
(c) 30.097 (d) 30.397

15. The value of $\frac{10\sqrt{6.25}}{\sqrt{6.25} - 0.5}$ is :

(a) 125 (b) 0.125
(c) 1.25 (d) 12.5

16. Simplify $\frac{\sqrt{25.4016} - \sqrt{1.0609}}{\sqrt{25.4016} + \sqrt{1.0609}}$:

(a) $\frac{401}{607}$ (b) $\frac{104}{706}$
(c) 41/76 (d) none of these

17. Let n be a positive integer. If $\frac{1}{n}$ has a terminating decimal expansion, then which one of the following is true?

(a) n is of the form 5^x, where x is a positive integer
(b) n is of the form 2^y, where y is non-positive integer
(c) n is of the form $2^x \cdot 5^y$ for some non-negative integers x and y
(d) n is of the form 10^z for some positive integer z

1.8 Indices and Surds

In the previous articles we have studied the multipli- cation, as $2 \times 2 = 4, 2 \times 2 \times 2 = 8, 2 \times 2 \times 2 \times 2 = 16, \dots$

Now, here we extend our discussion to the higher power or index of a number, as following

as
$$2 \times 2 = 2^2 = 4$$
$$2 \times 2 \times 2 = 2^3 = 8$$
$$2 \times 2 \times 2 \times 2 = 2^4 = 16$$
$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$$

Similarly, $2 \times 2 \times 2 \times \dots$ upto n times $= 2^n$

Thus
$$\underbrace{a \times a \times a \times a \times a \dots a}_{n \text{ factors}} = a^n$$

Where a is any number and n is a natural number. *i.e.,* a^n is continued product of n equal quantities, each equal to a and is called the nth power of a. 'n' is called the **index** or **exponent** and 'a' is called the **base** of a^n.

Therefore a^n is the exponential expression. a^n is read as 'a raised to the power n' or 'a to the power n'

Laws of Indices

If m and n are positive integers, then

1. $a^m \times a^n = a^{m+n}$ **2.** $\frac{a^m}{a^n} = a^{m-n}$; $(a \neq 0, m \geq n)$

3. $(ab)^n = a^n \cdot b^n$ **4.** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$; $(b \neq 0)$

5. $(a^m)^n = a^{m \cdot n}$ **6.** $(a^0) = 1$; $(a \neq 0)$

Some Important Results

1. If $a^x = k$, then $a = (k)^{1/x}$

2. If $a^{1/x} = k$, then $a = k^x$

3. If $a^x = b^y$, then $a = (b)^{y/x}$ and $b = (a)^{x/y}$

4. If $a^x = a^y$, then $x = y$, where $a \neq 0, 1$

5. $a^{-n} = 1/a^n$ and $a^n = 1/a^{-n}$

6. $a^{b^c} \neq (a^b)^c$; $b \neq c$

7. HCF : The HCF of $(a^m - 1)$ and $(a^n - 1)$ is equal to the $(a^{\text{HCF of } m, n} - 1)$

Exp. 1) Solve the following :

(i) $(5)^3$ (ii) $(-6)^4$ (iii) $(-2)^5$ (iv) $(4)^3$ (v) $(-4)^3$

Solution (i) $5^3 = 5 \times 5 \times 5 = 125$

(ii) $(-6)^4 = (-6) \times (-6) \times (-6) \times (-6) = 1296$

(iii) $(-2)^5 = (-2) \times (-2) \times (-2) \times (-2) \times (-2) = -32$

(iv) $(4)^3 = 4 \times 4 \times 4 = 64$

(v) $(-4)^3 = (-4) \times (-4) \times (-4) = -64$

Exp. 2) Solve the following expressions.

(i) $(-2)^3 \times 5^2$ (ii) $(-4)^3 \times 5^2 \times 7^0$

(iii) $(32)^{-1/5}$ (iv) $(243)^{4/5}$

(v) $(36)^{1/6}$ (vi) $\left(-\frac{1}{343}\right)^{-2/3}$

(vii) $3^{-3} + (-3)^3$ (viii) $(2^2 + 2^3 + 2^{-2} + 2^{-3})$

(ix) $2^{2x-1} = \frac{1}{8^{(x-3)}}$, then $x = ?$

(x) $4^{2x+1} = 8^{x+3}$ then $x = ?$

(xi) $3^{x-1} + 3^{x+1} = 90$, then $x = ?$

Solution (i) $(-2)^3 \times 5^2 = -8 \times 25 = -200$

(ii) $(-4)^3 \times 5^2 \times 7^0 = -64 \times 25 \times 1 = -1600$

(iii) $(32)^{-1/5} = (2^5)^{-1/5} = 2^{5 \times -\frac{1}{5}} = 2^{-1} = \frac{1}{2}$

(iv) $(243)^{4/5} = (3^5)^{4/5} = 3^{5 \times \frac{4}{5}} = 3^4 = 81$

(v) $(36)^{1/6} = (6^2)^{1/6} = 6^{2 \times \frac{1}{6}} = 6^{\frac{1}{3}}$

(vi) $\left(-\frac{1}{343}\right)^{-2/3} = \left(-\frac{1}{7^3}\right)^{-2/3} = (-7^{-3})^{-2/3}$

$= (-7)^{-3 \times -2/3} = (-7)^2 = 49$

(vii) $3^{-3} + (-3)^3 = \frac{1}{3^3} + (-3)^3 = \frac{1}{27} - 27$

$= \frac{1-729}{27} = -\frac{728}{27}$

(viii) $(2^2 + 2^3 + 2^{-2} + 2^{-3}) = \left(4 + 8 + \frac{1}{4} + \frac{1}{8}\right)$

$= 12 + \frac{3}{8} = \frac{99}{8}$

(ix) $2^{2x-1} = \frac{1}{8^{x-3}} \Rightarrow 2^{2x-1} = \frac{1}{2^{3(x-3)}}$

$\Rightarrow 2^{2x-1} = \frac{1}{2^{3x-9}}$

$\Rightarrow (2^{2x-1})(2^{3x-9}) = 1$

$\Rightarrow 2^{(2x-1)+(3x-9)} = 1$

$\Rightarrow 2^{5x-10} = 1 \Rightarrow 2^{5(x-2)} = 1$

$\Rightarrow 2^{5(x-2)} = 2^0$

$\Rightarrow 5(x-2) = 0$

$\Rightarrow x - 2 = 0 \Rightarrow x = 2$

(x) $4^{2x+1} = 8^{x+3}$

$\Rightarrow (2^2)^{(2x+1)} = (2^3)^{(x+3)}$

$\Rightarrow 2^{4x+2} = 2^{3x+9}$

$\Rightarrow 4x + 2 = 3x + 9$ ($\because$ base in both side is same)

$\Rightarrow x = 7$

(xi) $3^{x-1} + 3^{x+1} = 90$

$\Rightarrow \frac{3^x}{3} + 3.3^x = 90$

$\Rightarrow 3^x + 3^2 . 3^x = 90 \times 3$

$\Rightarrow 3^x(1 + 3^2) = 270$

$\Rightarrow 3^x(10) = 270 \Rightarrow 3^x = 27$

$\Rightarrow 3^x = 3^3 \Rightarrow x = 3$

Exp. 3) Solve the followings :

(i) $\left[\left(x + \frac{1}{y}\right)^a \left(x - \frac{1}{y}\right)^b\right] \div \left[\left(y + \frac{1}{x}\right)^a \left(y - \frac{1}{x}\right)^b\right]$ is equal to

(a) $\left(\frac{x}{y}\right)^{a+b}$ (b) $\left(\frac{y}{x}\right)^{(a+b)}$ (c) $\frac{x^a}{y^b}$ (d) $(xy)^{a+b}$

(ii) $\left(\frac{x^b}{x^c}\right)^a \times \left(\frac{x^c}{x^a}\right)^b \times \left(\frac{x^a}{x^b}\right)^c$ is equal to :

(a) 0 (b) 1

(c) abc (d) none of these

(iii) If $x^{x^{3/2}} = (x^{3/2})^x$, then the value of x is :

(a) $\frac{3}{2}$ (b) $\frac{9}{4}$ (c) $\frac{16}{25}$ (d) $\frac{8}{27}$

(iv) If $x^a = y^b = z^c$ and $y^2 = zx$ then the value of $\frac{1}{a} + \frac{1}{c}$ is :

(a) $\frac{b}{2}$ (b) $\frac{c}{2}$ (c) $\frac{2}{b}$ (d) $2a$

(v) $(a^{m-n})^l \times (a^{n-l})^m \times (a^{l-m})^n$

(a) 1 (b) 0 (c) 2 (d) a^{lmn}

(vi) If $a^x = b$, $b^y = c$ and $c^z = a$, then the value of xyz is

(a) 0 (b) 1 (c) $x + y + z$ (d) abc

(vii) $\left(\frac{x^a}{x^b}\right)^{a+b} \times \left(\frac{x^b}{x^c}\right)^{b+c} \times \left(\frac{x^c}{x^a}\right)^{c+a}$ is equal to :

(a) 1 (b) 0

(c) 2 (d) none of these

(viii) The value of

$$\frac{1}{1 + x^{b-a} + x^{c-a}} + \frac{1}{1 + x^{c-b} + x^{a-b}} + \frac{1}{1 + x^{a-c} + x^{b-c}}$$ is :

(a) 0 (b) x^{abc} (c) 1 (d) $x^{(a+b+c)}$

(ix) $2^{2^x} = 16^{2^{3x}}$, then x is equal to

(a) -1 (b) 0

(c) 1 (d) none of these

(x) The value of the expression

$$\frac{4^n \times 20^{m-1} \times 12^{m-n} \times 15^{m+n-2}}{16^m \times 5^{2m+n} \times 9^{m-1}} \text{ is :}$$

(a) 500 (b) 1 (c) 200 (d) $\frac{1}{500}$

(xi) The value of the expression

$$\frac{(0.3)^{1/3} \cdot \left(\frac{1}{27}\right)^{1/4} \cdot (9)^{1/6} \cdot (0.81)^{2/3}}{(0.9)^{2/3} \cdot (3)^{-1/2} \cdot \left(\frac{1}{3}\right)^{-2} \cdot (243)^{-1/4}} \text{ is :}$$

(a) 0.3 (b) 0.9 (c) 1.27 (d) 0.09

(xii) The value of expression $\dfrac{(0.6)^0 - (0.1)^{-1}}{\left(\frac{3}{2^3}\right)^{-1} \cdot \left(\frac{3}{2}\right)^3 + \left(-\frac{1}{3}\right)^{-1}}$ is :

(a) $-\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $\frac{9}{4}$

(xiii) If $\dfrac{9^n \times 3^2 \times (3^{-n/2})^{-2} - (27)^n}{3^{3m} \times 2^3} = \dfrac{1}{27}$

then the value of $(m - n)$ is :

(a) -1 (b) 1 (c) 2 (d) -2

Solution (i) $\left[\left(x + \frac{1}{y}\right)^a \left(x - \frac{1}{y}\right)^b\right] \div \left[\left(y + \frac{1}{x}\right)^a \left(y - \frac{1}{x}\right)^b\right]$

$$= \frac{\left(x + \frac{1}{y}\right)^a \left(x - \frac{1}{y}\right)^b}{\left(y + \frac{1}{x}\right)^a \left(y - \frac{1}{x}\right)^b} = \frac{\left(\frac{xy+1}{y}\right)^a \left(\frac{xy-1}{y}\right)^b}{\left(\frac{xy+1}{x}\right)^a \left(\frac{xy-1}{x}\right)^b}$$

$$= \frac{\frac{(xy+1)^a}{y^a} \cdot \frac{(xy-1)^b}{y^b}}{\frac{(xy+1)^a}{x^a} \cdot \frac{(xy-1)^b}{x^b}}$$

$$= \frac{(xy+1)^a (xy-1)^b}{y^{(a+b)}} \times \frac{x^{(a+b)}}{(xy+1)^a (xy-1)^b}$$

$$= \frac{x^{a+b}}{y^{a+b}} = \left(\frac{x}{y}\right)^{a+b}$$

Hence (a) is the correct option.

(ii) $\left(\frac{x^b}{x^c}\right)^a \times \left(\frac{x^c}{x^a}\right)^b \times \left(\frac{x^a}{x^b}\right)^c = (x^{b-c})^a \times (x^{c-a})^b \times (x^{a-b})^c$

$$= x^{(ab-ac)} \times x^{(bc-ab)} \times x^{(ac-bc)}$$

$$= x^{(ab-ac)+(bc-ab)+(ac-bc)} = x^0 = 1$$

Hence (b) is the correct option.

(iii) $x^{x^{3/2}} = (x^{3/2})^x \Rightarrow x^{x^{3/2}} = x^{(3/2)x}$

$\Rightarrow \quad x^{3/2} = \frac{3}{2}x \quad \Rightarrow x^{1/2} = \frac{3}{2} \quad \Rightarrow \quad x = \frac{9}{4}$

Hence (b) is the correct option.

(iv) If $x^a = y^b = z^c$ and $y^2 = zx$

Let $x^a = y^b = z^c = k$

$\Rightarrow \quad x = k^{1/a}, y = k^{1/b}, z = k^{1/c}$

Now, $\because \quad y^2 = zx$

$\therefore \quad (k^{1/b})^2 = (k^{1/c}) \cdot (k^{1/a})$

$\Rightarrow \quad k^{2/b} = k^{\frac{1}{c} + \frac{1}{a}}$

$\Rightarrow \quad \frac{1}{a} + \frac{1}{c} = \frac{2}{b}$

Hence (c) is the correct option.

(v) $(a^{m-n})^l \times (a^{n-l})^m \times (a^{l-m})^n = a^{ml-nl} \times a^{nm-lm} \times a^{ln-mn}$

$$= a^{ml-nl+nm-lm+ln-nm}$$

$$= a^0 = 1$$

Thus (a) is correct option.

(vi) If $a^x = b$, $b^y = c$, $c^z = a$

$\because \quad a^x = b \ \therefore \ (c^z)^x = b$

$\Rightarrow \quad c^{zx} = b \quad \Rightarrow \quad (b^y)^{zx} = b$

$\Rightarrow \quad b^{xyz} = b \quad \Rightarrow \quad xyz = 1$

Hence (b) is the correct option.

(vii) $\left(\frac{x^a}{x^b}\right)^{a+b} \times \left(\frac{x^b}{x^c}\right)^{b+c} \times \left(\frac{x^c}{x^a}\right)^{c+a}$

$$= (x^{a-b})^{(a+b)} \times (x^{b-c})^{(b+c)} \times (x^{c-a})^{(c+a)}$$

$$= x^{(a-b)(a+b)} \times x^{(b-c)(b+c)} \times x^{(c-a)(c+a)}$$

$$= x^{a^2-b^2} \times x^{b^2-c^2} \times x^{c^2-a^2}$$

$$= x^{a^2-b^2+b^2-c^2+c^2-a^2} = x^0 = 1$$

Thus (a) is the correct option.

(viii) $\dfrac{1}{1 + x^{b-a} + x^{c-a}} + \dfrac{1}{1 + x^{c-b} + x^{a-b}} + \dfrac{1}{1 + x^{a-c} + x^{b-c}}$

$$= \frac{1}{1 + \frac{x^b}{x^a} + \frac{x^c}{x^a}} + \frac{1}{1 + \frac{x^c}{x^b} + \frac{x^a}{x^b}} + \frac{1}{1 + \frac{x^a}{x^c} + \frac{x^b}{x^c}}$$

$$= \frac{x^a}{x^a + x^b + x^c} + \frac{x^b}{x^b + x^c + x^a} + \frac{x^c}{x^c + x^a + x^b}$$

$$= \frac{x^a + x^b + x^c}{x^a + x^b + x^c} = 1$$

Hence (c) is correct option.

(ix) $2^{2^x} = 16^{2^{3x}}$

$\Rightarrow \quad 2^{2^x} = (2^4)^{2^{3x}}$

$\Rightarrow \quad 2^{2^x} = 2^{4 \cdot 2^{3x}} = 2^{2^{3x+2}}$

$\Rightarrow \quad 2^{2^x} = 2^{2^{3x+2}}$ [Since base in both sides is equal]

$\Rightarrow \quad 2^x = 2^{3x+2}$

$\Rightarrow \quad x = 3x + 2$

$\Rightarrow \quad 2x = -2 \quad \Rightarrow x = -1$

Hence (a) is correct option.

(x) $\dfrac{4^n \times 20^{m-1} \times 12^{m-n} \times 15^{m+n-2}}{16^m \times 5^{2m+n} \times 9^{m-1}}$

$$= \frac{2^{2n} \times 2^{2m-2} \times 5^{m-1} \times 2^{2m-2n} \times 3^{m-n} \times 3^{m+n-2} \times 5^{m+n-2}}{2^{4m} \times 5^{2m+n} \times 3^{2m-2}}$$

$$= 2^{2n+2m-2+2m-2n-4m} \times 3^{m-n+m+n-2-2m+2} \times 5^{m-1+m+n-2-2m-n}$$

$$= 2^{-2} \times 3^0 \times 5^{-3} = \frac{1}{4} \times \frac{1}{125} = \frac{1}{500}$$

Hence (d) is the correct option.

(xi) $\dfrac{(0.3)^{1/3} \cdot \left(\frac{1}{27}\right)^{1/4} \cdot (9)^{1/6} \cdot (0.81)^{2/3}}{(0.9)^{2/3} \cdot (3)^{-1/2} \cdot \left(\frac{1}{3}\right)^{-2} \cdot (243)^{-1/4}}$

$$= \frac{\left(\frac{3}{10}\right)^{1/3} \left(\frac{1}{3}\right)^{3/4} (3)^{1/3} \cdot \left(\frac{81}{100}\right)^{2/3}}{\left(\frac{9}{10}\right)^{2/3} \left(\frac{1}{3}\right)^{1/2} \cdot (3)^2 \, (3)^{-5/4}}$$

$$= \frac{3^{\frac{1}{3}+\frac{1}{3}-\frac{3}{4}+\frac{8}{3}} \times 10^{-\frac{1}{3}-\frac{4}{3}}}{3^{\frac{4}{3}+2-\frac{1}{2}-\frac{5}{4}} \times 10^{-\frac{2}{3}}}$$

$$= \frac{3^{31/12} \times 10^{-\frac{5}{3}}}{3^{\frac{19}{12}} \times 10^{-\frac{2}{3}}} = 3 \times 10^{-1} = \frac{3}{10} = 0.3$$

(xii) $\dfrac{(0.6)^0 - (0.1)^{-1}}{\left(\frac{3}{2^3}\right)^{-1} \cdot \left(\frac{3}{2}\right)^3 + \left(-\frac{1}{3}\right)^{-1}}$

$$= \frac{1 - \left(\frac{1}{10}\right)^{-1}}{(3^{-1})\,(2^3) \times 3^3 \times 2^{-3} + (-3)^1}$$

$$= \frac{1-10}{3^{3-1} \cdot 2^0 - 3} = \frac{-9}{9-3} = \frac{-9}{6} = \frac{-3}{2}$$

Hence (a) is the correct option.

(xiii) $\dfrac{9^n \times 3^2 \times (3^{-n/2})^{-2} - (27)^n}{3^{3m} \times 2^3} = \dfrac{1}{27}$

$$\Rightarrow \frac{3^{2n} \times 3^2 \times 3^n - 3^{3n}}{3^{3m} \times 2^3} = \frac{1}{3^3}$$

$$\Rightarrow \frac{3^{2n+2+n} - 3^{3n}}{8 \times 3^{3m}} = \frac{1}{3^3}$$

$$\Rightarrow \frac{3^{3n+2} - 3^{3n}}{3^{3m} \times 8} = 3^{-3}$$

$$\Rightarrow \frac{3^{3n}\,(3^2 - 1)}{3^{3m} \times 8} = 3^{-3}$$

$$\Rightarrow 3^{3n-3m} = 3^{-3} \Rightarrow 3n - 3m = -3 \Rightarrow m - n = 1$$

Hence (b) is the correct option.

Exp. 4) Find the HCF of $(2^{315} - 1)$ and $(2^{25} - 1)$ is

(a) 5
(b) 2
(c) 31
(d) none of the above

Solution The required HCF $= (2)^{\text{HCF of } 315,\,25} - 1 = 2^5 - 1 = 31$

Hence (c) is the correct option.

Surds

In the previous articles we have studied about the square roots and cube roots etc. There we have studied those numbers whose roots can be found in the form of an integer, but here we study only those numbers whose roots are not the integers.

Surds : When a root of a rational number (*i.e.,* quantities of the type $\sqrt[n]{a}$, a being a rational number) can not be exactly obtained, then this root is called a **surd.**

For Example $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}$, etc.

$$\sqrt[3]{2}, \sqrt[3]{3}, \sqrt[3]{4}, \sqrt[3]{5}, \sqrt[3]{6}, \sqrt[3]{7}, \sqrt[3]{9}, \sqrt[3]{10}, \text{...etc}$$

$$\sqrt[4]{2}, \sqrt[4]{3}, \sqrt[4]{4}, \sqrt[4]{5}, \sqrt[4]{6}, \sqrt[4]{7}, \text{... etc.}$$

But $\sqrt{4} = 2, \sqrt[3]{27}, \sqrt[4]{16}, \sqrt[5]{32}, \sqrt[6]{729}$,... etc are not the surds.

Thus it can be said that ''all surds are irrational numbers, but all irrational numbers are not the surds''.

For example π and e are irrational, but not surds.

Order of Surds : In a surd $\sqrt[n]{a}$ $[= (a)^{1/n}]$, the value of n is called the order of the surd and a is called the **radicand.**

The surds of second order are called quadratic surds *e.g.,* $\sqrt{5}, \sqrt{6}, (7)^{3/2}$. The surds of third order are called cubic surds *e.g.,* $\sqrt[3]{3}, \sqrt[3]{7}, (6)^{5/3}, (9)^{2/3}$. The surds of fourth order are called biquadratic surds *e.g.,* $\sqrt[4]{2}, \sqrt[4]{3}, \sqrt[4]{7}, (5)^{7/4}$.

Mixed Surds : The product of a rational number with a surd is known as a **mixed surd.**

For example, $3 \cdot \sqrt{2}, 4\sqrt{7}, 5 \cdot \sqrt[3]{6}, \sqrt{18}\ (= 3\sqrt{2})$

Pure Surd : A surd having no rational factor is known as pure surd.

e.g., $\sqrt[4]{2}, \sqrt[3]{7}, \sqrt[5]{8}$ etc.

Similar Surds : If the radicands of two or more rationalised surds are same, then these surds are called the **similar surds.** For example, $\sqrt{5}, 3\sqrt{5}, \sqrt{20}\ (= 2\sqrt{5})$ and $\sqrt{80}\ (= 4\sqrt{5})$ etc.

Conjugate or Complementary Surds : For every surd of the form $\sqrt{a} + \sqrt{b}$ there exists another surd $\sqrt{a} - \sqrt{b}$, which is called the conjugate surd of previous surd and *vice-versa.*

For example,
$\{(2\sqrt{3}+5\sqrt{3})$ and $(2\sqrt{3}-5\sqrt{3})\}$, $\{(\sqrt{7}+\sqrt{8})$ and $(\sqrt{7}-\sqrt{8})\}, \ldots$ etc are the pairs of conjugate surds.

Properties of Surds

1. A quadratic surd cannot be equal to the sum or difference of a rational number and a quadratic surd *e.g.,* $\sqrt{a}+b \neq \sqrt{c}$ or $\sqrt{a}-b \neq \sqrt{c}$, where $\sqrt{a}, \sqrt{c}$ are quadratic surds and b is a rational number.
2. The product and quotient of two dissimilar quadratic surds can not be rational *e.g.,* $\sqrt{2}\,.\,\sqrt{5}=\sqrt{10}$ or $\dfrac{\sqrt{6}}{\sqrt{2}}=\sqrt{3}$
3. If $a+\sqrt{b}=c+\sqrt{d}$ or $a-\sqrt{b}=c-\sqrt{d}$, then $a=c$ and $b=d$.
4. If $a \pm \sqrt{b}=0$ then $a=b=0$
5. If $a+\sqrt{b}=c+\sqrt{d}$, then $a-\sqrt{b}=c-\sqrt{d}$ and its *vice-versa.*
6. If $\sqrt{a+\sqrt{b}}=\sqrt{c}+\sqrt{d}$, then $\sqrt{a-\sqrt{b}}=\sqrt{c}-\sqrt{d}$ and its *vice-versa.*

Addition : Only similar surds can be simplified

e.g., $2\sqrt{3}+7\sqrt{3}=9\sqrt{3}$

and $8\sqrt{5}+3\sqrt{5}+\sqrt{20}=8\sqrt{5}+3\sqrt{5}+2\sqrt{5}=13\sqrt{5}$

Subtraction : Only similar surds can be simplified

e.g., $7\sqrt{3}-2\sqrt{3}=5\sqrt{3}$

and $8\sqrt{5}-3\sqrt{5}=5\sqrt{5}$ etc.

NOTE $8\sqrt{5} \pm 2\sqrt{6}$ can not be simplified, because radicands are different.

Multiplication : $3^{1/4} \times 6^{1/2} \times 9^{1/3}$

$$= 3^{3/12} \times 6^{6/12} \times 9^{4/12}$$
$$= (3^3 \times 6^6 \times 9^4)^{1/12}$$
$$= (3^3 \times 2^6 \times 3^6 \times 3^8)^{1/12}$$
$$= (3^{17} \times 2^6)^{1/12}$$

Division : $25 \div \sqrt{5} = \dfrac{25}{\sqrt{5}} = \dfrac{(5)^2}{(5)^{1/2}} = 5^{2-\frac{1}{2}} = 5^{3/2} = \sqrt{125}$

Rationalisation : Making two or more surds a rational number by multiplication, is called the rationalisation

e.g., $3\sqrt{2} \times 4\sqrt{2} = 24$

$$7\sqrt{5} \times \sqrt{5} = 35$$

$$(9\sqrt{3}-5\sqrt{3})\,(9\sqrt{3}+5\sqrt{3}) = (9\sqrt{3})^2 - (5\sqrt{3})^2 = 168$$

Laws of Surds

(i) $\sqrt[n]{a\,.\,b} = \sqrt[n]{a}\,.\,\sqrt[n]{b}$ (ii) $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

(iii) $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$ (iv) $(\sqrt[n]{a})^n = (a^{1/n})^n = a$

(v) $\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$

Exp. 1) Which one is smaller out of $\sqrt[3]{2}$ and $\sqrt[4]{3}$?

Solution $\sqrt[3]{2} = (2)^{1/3} = 2^{4/12} = (16)^{1/12}$

and $\sqrt[4]{3} = (3)^{1/4} = 3^{3/12} = (27)^{1/12}$

(Taking the LCM of surd)

Hence $\sqrt[3]{2} < \sqrt[4]{3}$

Exp. 2) Which one is greatest out of $\sqrt[3]{5}, \sqrt[4]{3}, \sqrt[5]{4}$?

Solution $\sqrt[3]{5} = (5)^{1/3} = (5)^{20/60} = (5^{20})^{1/60} = \sqrt[60]{5^{20}}$

$\sqrt[4]{3} = (3)^{1/4} = (3)^{15/60} = (3^{15})^{1/60} = \sqrt[60]{3^{15}}$

$\sqrt[5]{4} = (4)^{1/5} = (4)^{12/60} = (4^{12})^{1/60} = \sqrt[60]{4^{12}}$

Obviously $\sqrt[3]{5}$ is the greatest.

Exp. 3) If $p=\sqrt[3]{5}, q=\sqrt[4]{3}, r=\sqrt[5]{4}$ then the correct relationship is :

(a) $p > q > r$ (b) $q < r < p$
(c) $r > p > q$ (d) can't be determined

Solution In the previous question we have found that p is the greatest.

Again $q=\sqrt[60]{3^{15}}$ and $r=\sqrt[60]{4^{12}}$

Now $3^{15} \gtreqless 4^{12}$

[Here the sign signifies the trichotomy- relation *i.e.,* if $x \gtreqless y$, then either $x<y$ or $x=y$ or $x>y$]

$\Rightarrow 3^{12}\,.\,3^3 \gtreqless 4^{12}$

$\Rightarrow 3^3 \gtreqless \dfrac{4^{12}}{3^{12}}$

$\Rightarrow 3^3 \gtreqless \left(\dfrac{4}{3}\right)^{12}$

$\Rightarrow 3^3 \gtreqless (1.33)^{12}$

$\Rightarrow 3^3 \gtreqless [(1.33)^4]^3 \Rightarrow 3^3 < (3.16)^3$

Thus $q < r$

Hence $p > r > q$, so (b) is the correct option.

Exp. 4) If $5\sqrt{5} \times 5^3 \div 5^{-3/2} = 5^{a+2}$ then the value of a is :

(a) 4 (b) 5 (c) 9 (d) 16

Solution $\dfrac{5\sqrt{5} \times 5^3}{5^{-3/2}} = 5^{a+2}$

$\Rightarrow \dfrac{5^{4+\frac{1}{2}}}{5^{-\frac{3}{2}}} = 5^{a+2} \Rightarrow 5^{4+\frac{1}{2}+\frac{3}{2}} = 5^{a+2}$

$\Rightarrow 5^6 = 5^{a+2}$

$\Rightarrow 6 = a+2 \Rightarrow a = 4$

Hence (a) is the correct option.

Exp. 5) Find the value of $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$.

Solution $\dfrac{2+\sqrt{3}}{2-\sqrt{3}} = \dfrac{2+\sqrt{3}}{2-\sqrt{3}} \times \dfrac{2+\sqrt{3}}{2+\sqrt{3}}$

(Multiplying numerator and denominator by conjugate)

$$= \frac{(2+\sqrt{3})^2}{4-3} = \frac{4+3+4\sqrt{3}}{1}$$

$$= 7+4\sqrt{3}$$

Exp. 6) The value of $\dfrac{\sqrt{5}-2}{\sqrt{5}+2}$ is :

(a) $4-9\sqrt{5}$ (b) $9+4\sqrt{5}$

(c) $9-4\sqrt{5}$ (d) $7+4\sqrt{5}$

Solution $\dfrac{\sqrt{5}-2}{\sqrt{5}+2} = \dfrac{(\sqrt{5}-2)}{(\sqrt{5}+2)} \times \dfrac{(\sqrt{5}-2)}{(\sqrt{5}-2)}$

$$= \frac{(\sqrt{5}-2)^2}{5-4} = \frac{5+4-4\sqrt{5}}{1} = 9-4\sqrt{5}$$

Thus (c) is the correct option.

Exp. 7) Find the value of $\dfrac{5}{6\sqrt{6}} \times \dfrac{12\sqrt{30}}{25\sqrt{5}}$.

Solution $\dfrac{5}{6\sqrt{6}} \times \dfrac{12\sqrt{30}}{25\sqrt{5}} = \dfrac{5}{6\sqrt{6}} \times \dfrac{12\sqrt{6}\ \sqrt{5}}{25\sqrt{5}} = \dfrac{2}{5}$

Exp. 8) Arrange the following in descending order

$$\sqrt{3}-\sqrt{2},\ \sqrt{4}-\sqrt{3},\ \sqrt{5}-\sqrt{4},\ \sqrt{2}-1.$$

Solution $\sqrt{3}-\sqrt{2} = \dfrac{\sqrt{3}-\sqrt{2}}{1} \times \dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

$$= \frac{3-2}{\sqrt{3}+\sqrt{2}} = \frac{1}{\sqrt{3}+\sqrt{2}}$$

Similarly, $\sqrt{4}-\sqrt{3} = \dfrac{1}{\sqrt{4}+\sqrt{3}}$, $\sqrt{5}-\sqrt{4} = \dfrac{1}{\sqrt{5}+\sqrt{4}}$

and $\sqrt{2}-1 = \dfrac{1}{\sqrt{2}+1}$

As we know, if the numerator is same then the fraction whose denominator is larger the fraction will be lower.

Hence the correct order of descending number is

$$(\sqrt{2}-1) > (\sqrt{3}-\sqrt{2}) > (\sqrt{4}-\sqrt{3}) > (\sqrt{5}-\sqrt{4})$$

Exp. 9) $\sqrt[r]{\dfrac{9^{\left(r+\frac{1}{4}\right)}\sqrt{3\,.\,3^{-r}}}{3\,.\,\sqrt{3^{-r}}}} = k$, then the value of k is

(a) 3 (b) 3^2 (c) 3^3 (d) $\sqrt[r]{3}$

Solution $\sqrt[r]{\dfrac{3^{2r+\frac{1}{2}}\,3^{\left(\frac{1-r}{2}\right)}}{3^{1-\frac{r}{2}}}} = \sqrt[r]{\dfrac{3^{\frac{4r+1+1-r}{2}}}{3^{\frac{2-r}{2}}}} = \sqrt[r]{3^{\left(\frac{3r+2}{2}\right)-\left(\frac{2-r}{2}\right)}}$

$$= \sqrt[r]{3^{2r}} = 3^{\frac{2r}{r}} = 3^2$$

Hence (b) is the correct option.

Exp. 10) If m, n are the positive integers $(n>1)$ such that $m^n = 121$, then value of $(m-1)^{n+1}$ is :

(a) 12321 (b) 1

(c) 1000 (d) 11

Solution $\because$ $m^n = 121$ and $n>1$

$\therefore$ $(11)^2 = 121$

$\Rightarrow$ $m = 11$ and $n = 2$

Hence $(m-1)^{n+1} = (10)^3 = 1000$

Thus (c) is the correct option.

Exp. 11) Simplify : $\sqrt{192} - \dfrac{1}{2}\sqrt{48} - \sqrt{75}$

Solution $\sqrt{192} - \dfrac{1}{2}\sqrt{48} - \sqrt{75} = 8\sqrt{3} - \dfrac{4}{2}\sqrt{3} - 5\sqrt{3}$

$$= 8\sqrt{3} - 2\sqrt{3} - 5\sqrt{3} = \sqrt{3}$$

Exp. 12) Find the square root of $7-2\sqrt{10}$.

Solution $7-2\sqrt{10} = 5+2-2\sqrt{5\times 2}$

$\Rightarrow$ $7-2\sqrt{10} = (\sqrt{5})^2 + (\sqrt{2})^2 - 2\sqrt{5}\,.\,\sqrt{2}$

$\Rightarrow$ $7-2\sqrt{10} = (\sqrt{5}-\sqrt{2})^2$

Thus the $\sqrt{7-2\sqrt{10}} = \pm(\sqrt{5}-\sqrt{2})$

Exp. 13) Find the value of $\left(\dfrac{1}{\sqrt{5}-2} + \dfrac{1}{\sqrt{5}+2}\right)^2$.

Solution $\left(\dfrac{1}{\sqrt{5}-2} + \dfrac{1}{\sqrt{5}+2}\right)^2 = \left[\dfrac{\sqrt{5}+2+\sqrt{5}-2}{(\sqrt{5}-2)(\sqrt{5}+2)}\right]^2$

$$= \left[\frac{2\sqrt{5}}{(\sqrt{5})^2-(2)^2}\right]^2 = \frac{4\times 5}{5-4} = \frac{20}{1} = 20$$

Introductory Exercise 1.8

1. In the equation $4^{x+2} = 2^{x+3} + 48$, the value of x will be
(a) $-\frac{3}{2}$ (b) -2
(c) -3 (d) 1

2. Consider the following statements :
Assertion (A) $a^0 = 1, a \neq 0$
Reason (R) $a^m \div a^n = a^{m-n}$, m, n being integers.
Of these statements :
(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true and R is not the correct explanation of A
(c) A is true, but R is false
(d) A is false, but R is true

3. If $3^n = 27$ then 3^{n-2} is :
(a) 3 (b) 1/3
(c) 1/9 (d) 9

4. If $4^{x+3} \times 2^{x-3} - 128 = 0$ then the value of x is :
(a) $\frac{1}{3}$ (b) $\frac{2}{3}$
(c) $\frac{3}{3}$ (d) $\frac{4}{3}$

5. If $a^m \cdot a^n = a^{mn}$, then $m(n-2) + n(m-2)$ is :
(a) 0 (b) 1
(c) -1 (d) $\frac{1}{2}$

6. $\frac{6^6 + 6^6 + 6^6 + 6^6 + 6^6 + 6^6}{3^6 + 3^6 + 3^6} \div \frac{4^6 + 4^6 + 4^6 + 4^6}{2^6 + 2^6} = 2^n$,
then the value of n is :
(a) -1 (b) 0
(c) $\frac{1}{2}$ (d) 1

7. If $x^y = y^x$ and $y = 2x$, then x is equal to :
(a) 2 (b) -2
(c) 1 (d) -1

8. If $2^{x+3} \cdot 4^{2x-5} = 2^{3x+7}$, then the value of x is :
(a) 3 (b) 4
(c) 6 (d) 7

9. $9^{3/2} \div (243)^{-2/3}$ simplifies to :
(a) $3^{10/3}$ (b) $3^{19/3}$
(c) $3^{1/3}$ (d) 3^{19}

10. The solution of $(25)^{x-2} = (125)^{2x-4}$:
(a) 3/4 (b) 0
(c) 2 (d) -2

11. The value of
$\left(\frac{x^a}{x^b}\right)^{(a^2+ab+b^2)} \left(\frac{x^b}{x^c}\right)^{(b^2+bc+c^2)} \left(\frac{x^c}{x^a}\right)^{(c^2+ca+a^2)}$ is :
(a) -1 (b) x^{abc}
(c) 1 (d) $x^{(a+b+c)}$

12. If $a^x = b^y = c^z$ and $\frac{b}{a} = \frac{c}{b}$, then $\frac{2z}{x+z}$ is equal to :
(a) $\frac{y}{x}$ (b) $\frac{x}{y}$
(c) $\frac{x}{2}$ (d) $\frac{z}{x}$

13. If $a^{1/m} = b^{1/n} = c^{1/p}$ and $abc = 1$, then $m + n + p$ is equal to :
(a) 0 (b) 2
(c) 1 (d) -2

14. Value of $[(x^l)^{1-\frac{1}{l}}]^{\frac{1}{l-1}}$ is :
(a) x (b) 1
(c) x^{-1} (d) x^{-l}

15. Which one of the following sets of surds is in correct sequence of ascending order of their values?
(a) $\sqrt[4]{10}, \sqrt[3]{6}, \sqrt{3}$ (b) $\sqrt{3}, \sqrt[4]{10}, \sqrt[3]{6}$
(c) $\sqrt{3}, \sqrt[3]{6}, \sqrt[4]{10}$ (d) $\sqrt[4]{10}, \sqrt{3}, \sqrt[3]{6}$

16. If $x = \frac{1}{\sqrt{2} - 1}$, then the value of $x^2 - 6 + \frac{1}{x^2}$ is :
(a) 0 (b) 1
(c) $\sqrt{2}$ (d) can't be determined

17. The value of $\frac{3}{5\sqrt[4]{27}}$, when its denominator is a rational number :
(a) $\frac{\sqrt[2]{3}}{6}$ (b) $\frac{\sqrt[4]{3}}{5}$
(c) $\frac{\sqrt[3]{4}}{5}$ (d) none of these

18. The value of denominator when $\frac{7\sqrt{3} - 5\sqrt{2}}{\sqrt{18} + \sqrt{48}}$ is rationalised :
(a) 15 (b) 25 (c) 30 (d) 50

19. If $\sqrt{5 + \sqrt[3]{x}} = 3$, then the value of x is :
(a) 9 (b) 27
(c) 64 (d) 343

20. If $A = \sqrt{2}$, $B = \sqrt[3]{3}$ and $C = \sqrt[4]{4}$ then which of the relation is correct?
(a) $A > B > C$ (b) $A > B = C$
(c) $A = C < B$ (d) none of these

1.9 Factorials, Last Digits and Remainders

Factorial

The product of n consecutive natural numbers (or positive integers) starting from 1 to n is called as the factorial 'n'.

i.e., $n! = 1\times2\times3\times4\times5\times6\dots(n-2)(n-1)\,n$

So, the $4! = 1\times2\times3\times4 = 4\times3\times2\times1$

$5! = 1\times2\times3\times4\times5 = 5\times4\times3\times2\times1$

$6! = 1\times2\times3\times4\times5\times6 = 6\times5\times4\times3\times2\times1$

NOTE 1. Factorial n is written as "$n!$". 2. $0! = 1$ and $1! = 1$

Properties

1. $n!$ is always an even number if $n \geq 2$
2. $n!$ always ends with zero if $n \geq 5$

Exp. 1) If $m^n - m = (m-n)!$ where $m > n > 1$ and $m = n^2$, then the value of $m^2 + n^2$ is :

(a) 272 (b) 90
(c) 20 (d) none of (a), (b), (c)

Solution Let us consider $n = 2$ $(\because n > 1)$

Then $m^n - m = 4^2 - 4 = 12$ $(\because m = n^2)$

and $(m-n)! = (4-2)! = 2! = 2$

Hence it is impossible.

Again consider $n = 3$, then

$m^n - m = 9^3 - 9 = 720$

and $(m-n)! = (9-3)! = 6! = 720$

Thus we get $n = 3$ and $m = 9$ the probable values

Now $m^2 + n^2 = 9^2 + 3^2 = 81 + 9 = 90$

Hence (b) is the correct option.

Exp. 2) If $P + P! = P^3$, then the value of P is :

(a) 4 (b) 6 (c) 0 (d) 5

Solution Consider $P = 5$, then

$5 + 5! = 5^3$

$5 + 120 = 125$

$125 = 125$

Thus (d) is correct option.

Exp. 3) If $P + P! = P^2$, then the value of P is :

(a) 5 (b) 7 (c) 3 (d) 0

Solution Consider $P = 3$ then

$3 + 3! = 3^2$;

$\Rightarrow$ $3 + 6 = 9 \Rightarrow 9 = 9$

Hence (c) is the correct option.

Exp. 4) If $n! - n = n$, then the value of n is :

(a) 4 (b) 5 (c) 6 (d) 3

Solution Consider $n = 3$ then

$3! - 3 = 3$

$\Rightarrow$ $6 - 3 = 3$

$\Rightarrow$ $3 = 3$

Hence (d) is the correct answer.

Exp. 5) The appropriate value of P for the relation $(P!+1) = (P+1)^2$ is :

(a) 3 (b) 4
(c) 5 (d) none of these

Solution Let us consider $P = 4$ then

$4! + 1 = (4+1)^2$

$\Rightarrow$ $24 + 1 = 5^2$

$\Rightarrow$ $25 = 25$

Hence (b) is the correct option.

Exp. 6) The value of $8! \div 4!$ is :

(a) 4 (b) 16 (c) 2 (d) 1680

Solution $\dfrac{8!}{4!} = \dfrac{8.7.6.5.4.3.2.1}{4.3.2.1}$

$= 8.7.6.5 = 1680$

Hence (d) is the correct option.

Exp. 7) If $n! = \dfrac{(n+4)!}{(n+1)!}$, then the value of n is :

(a) 5 (b) 18 (c) 6 (d) 9

Solution $n! = \dfrac{(n+4)!}{(n+1)!}$

$\Rightarrow$ $n! = \dfrac{1.2.3....n.(n+1)(n+2)(n+3)(n+4)}{1.2.3...n.(n+1)}$

$\Rightarrow$ $n! = (n+2)(n+3)(n+4)$

$n! = (n+2)(n+3)(n+4)$

now $5! \neq 7\times8\times9$

and $18! \neq 20\times21\times22$

and $9! \neq 11\times12\times13$

but $6! = 8\times9\times10$

$720 = 720$

Hence $n = 6$ is the correct answer, thus (c) is the right choice.

Exp. 8) The value of

$(1.2.3\dots9).(11.12.13\dots19).(21.22.23\dots29).(31.32.33\dots39).$
$(41.42.43\dots49)(51.52.53\dots59)\dots(91.92\dots99)$

(a) $\dfrac{100!}{36288\times10^{11}}$ (b) $\dfrac{99!}{388\times10^{11}}$

(c) $\dfrac{99!}{36288\times10^{10}}$ (d) can't be determined

Solution $(1.2.3...9).(11.12.13...19).(21.22.23...29)\dots$
$\dots(91.92.93....99)$

$= (1.2.3...9)\dfrac{10}{10}(11.12.13...19)\dfrac{20}{20}$

$$(21\,.\,23\,.\,23\,...\,29)\frac{30}{30}...(91\,.\,92\,.\,93\,...\,99)$$

$$=\frac{1\,.\,2\,.\,3\,...\,99}{10\,.\,20\,.\,30\,...\,90}=\frac{99!}{362880\times 10^9}=\frac{99!}{36288\times 10^{10}}$$

Hence (c) is the correct option.

Exp. 9) The expression $1!+2!+3!+4!+\ldots+n!$ (where $n\geq 5$) is not a/an :

(a) composite number (b) odd number
(c) perfect square (d) multiple of 3

Solution $1!+2!+3!+4!+5!=1+2+6+24+120=153$

Since, the options (a), (b) and (d) are ruled out so the remaining option (c) is the correct one.

Exp. 10) The HCF and LCM of 13! and 31! are respectively :

(a) 12! and 32! (b) 13! and 31!
(c) 26 and 403 (d) can't be determined

Solution Since 13! is contained in 31! so the LCM is 31! and 13! is common in 13! and 31!, so the HCF is 13!

Number of zeros at the end of the product

We know that $10=5\times 2$, $100=5^2\times 2^2$, $1000=5^3\times 2^3$ etc.

So we can say that for n number of zeros at the end of the product we need exactly n combinations of ''5 × 2''. For example.

$2\times 3\times 5\times 7=210$. There is only one zero at the end of the product (or resultant value)

Again, $2\times 3\times 5\times 6\times 7\times 15=2\times 3\times 5\times 2\times 3\times 7\times 3\times 5$

$$=\underline{2\times 2}\times\underline{5\times 5}\times 3\times 3\times 3\times 7$$

$$=100\times 189=18900$$

Thus there are two zeros because there are two combinations of ''5 × 2''.

Now, $1\times 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9$

$$=1\times 2\times 3\times 2\times 2\times 5\times 2\times 3\times 7\times 2\times 2\times 2\times 3\times 3$$

$$=1\times 2\times 2\times 2\times 2\times 2\times 2\times 2\times 5\times 3\times 3\times 3\times 3\times 7$$

$$=1\times 2^7\times 5\times 3^4\times 7$$

$$=1\times 2^6\times 2\times 5\times 3^4\times 7$$

$$=10\times 2^6\times 3^4\times 7$$

$$=10\times 64\times 81\times 7$$

$$=10\times 36288=362880$$

Thus there is only one zero at the end of the product since there is only one combination of 5×2..

Again $4\times 125\times 3=2\times 2\times 5\times 5\times 5\times 3$

$$=2^2\times 5^3\times 3=2^2\times 5^2\times 5\times 3$$

$$=(2\times 5)^2\times 15$$

$$=100\times 15=1500$$

Thus there are only two zeros at the end of the product since there are only two combinations of ''5×2''.

NOTE The number of zeros at the end of the product depends upon 2×5, but the condition is that
(i) $2^k\times 5^l$ gives k number of zeros if $k<l$
(ii) $2^k\times 5^l$ gives l number of zeros if $l<k$

Exp. 1) Find the number of zeros in the product of 10!.

Solution $10!=1\times 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9\times 10$

It is obvious from the above expression that there are only two 5s and eight 2s.
Since the number of 5s are less (than the number of 2's) so the number of 5s will be effective to form the combination of '5 × 2'.
Thus there are only 2 zeros at the end of the product of 10!.

Exp. 2) Find the number of zeros at the end of the product of $2^{222}\times 5^{555}$.

Solution Since the number of 2s are less than the number of 5s hence the restriction is imposed by the number of 2s.
Thus there can be only 222 pairs of (5 × 2). Hence the number of zeros at the end of the product of the given expression will be 222.

Exp. 3) Find the number of zeros at the end of the product of the expression

$$2^1\times 5^2\times 2^3\times 5^4\times 2^5\times 5^6\times 2^7\times 5^8\times 2^9\times 5^{10}.$$

Solution Since the number of 2's are less than the number of 5's hence number of 2s will be effective. Thus

$$2^1\times 2^3\times 2^5\times 2^7\times 2^9\times 5^2\times 5^4\times 5^6\times 5^8\times 5^{10}$$

$$=2^{25}\times 5^{30}=2^{25}\times 5^{25}\times 5^5=(2\times 5)^{25}\times 5^5$$

$$=10^{25}\times 5^5=5^5\times 10^{25}$$

Thus there will be 25 zeros at the end of the product of the given expression.

Exp. 4) The number of zeros at the end of the expression $10+100+1000+\ldots+10000000000$ is :

(a) 1 (b) 10
(c) 55 (d) none of these

Solution

```
         10
        100
       1000
   .........
   .........
10000000000
11111111110
```

Thus there is only one zero at the end of resut.
Hence (a) is the correct option.

Exp. 5) The number of zeros at the end of the product of the expression $10 \times 100 \times 1000 \times 10000 \times \ldots 10000000000$ is :

(a) 10 (b) 100
(c) 50 (d) 55

Solution $10 \times 100 \times 1000 \times \ldots 10000000000$

$$= 10^1 \times 10^2 \times 10^3 \times \ldots \times 10^{10}$$
$$= 10^{(1+2+3+\ldots+10)} = 10^{55}$$

Hence (d) is the correct option.

Exp. 6) Number of zeros at the end of the following expression $(5!)^{5!} + (10!)^{10!} + (50!)^{50!} + (100!)^{100!}$ is :

(a) 165 (b) 120
(c) 125 (d) none of these

Solution The number of zeros at the end of $(5!)^{5!} = 120$

[$\because 5! = 120$ and thus $(120)^{120}$ will give 120 zeros]

and the number of zeros at the end of the $(10!)^{10!}$, $(50!)^{50!}$ and $(100!)^{100!}$ will be greater than 120. Now since the number of zeros at the end of the whole expression will depends on the number which has the least number of zeros at the end of the number among other given numbers. So, the number of zeros at the end of the given expression is 120.

Divisibility of a Factorial Number by the Largest Power of Any Number

Let us start with very simple example. As if we consider 1!, 2!, 3! or 4!, none of them is divisible by 5, because 5 is not the factor, involved in the given factorial number.

Now, if we consider any factorial number greater than 4!, every number consists of 5 or the higher powers of 5 in the factorial numbers. For example starting from 5! every number has 5 as its factor. That is

$$5! = 1 \times 2 \times 3 \times 4 \times 5$$
$$6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6$$
$$7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$10! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \text{ etc.}$$

Thus it is obvious that every number greater than 4! is divisible by 5. But, as we move towards higher number we find that the frequency of the occurrence of 5 (or any particular number) increases. As 5! contains only one 5, 10! contains two 5s, 15! contains three 5s, 20! contains four 5s but 25! contains suddenly six 5s instead of five 5s. So it becomes a very tedious work to calculate the occurrence of any particular number involved as a factor of any higher factorial number.

Similarly, we can find that how many times 3 is contained in the 30! or in other words what is the largest power of 3 that can divide 30!

Now, if we try to do it manually, we see that there are 10 multiples of 3 *viz.,* 3, 6, 9, 12, 15, 18, 21, 24, 27, 30. But, there are some multiples which contains 3 at least two times for Example 9), 18 and 27.

Again there are some multiples which contain 3 three times as in 27. Thus the factor 3 totally occurs $14\,(= 10 + 3 + 1)$ times in 30!.

Hence you can understand that how tedious the calculation could be if we have to find the highest power of 3 that can exactly divide 9235!

Therefore to make the calculation easier we can use the following formula.

Suppose we have to find the highest power of k that can exactly divided n!, we divide n by k, n by k^2, n by $k^3 \ldots$ and so on till we get $\left\lfloor \frac{n}{k^x} \right\rfloor$ equal to 1 (where, $[P]$ means the greatest integer less than or equal to P) and then add up as

$$\left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{n}{k^2} \right\rfloor + \left\lfloor \frac{n}{k^3} \right\rfloor + \left\lfloor \frac{n}{k^4} \right\rfloor + \ldots + \left\lfloor \frac{n}{k^x} \right\rfloor$$

NOTE For more information about greatest integer number (or function) *i.e.,* GIF refer the chapter "FUNCTIONS".

Exp. 1) Find the largest power of 5 contained in 124!

Solution $\left\lfloor \frac{124}{5} \right\rfloor + \left\lfloor \frac{124}{5^2} \right\rfloor = 24 + 4 = 28$

[We cannot do it further since 124 is not divisible by 5^3]

Hence, there are 28 times 5 involved as a factor in 124!

Exp. 2) Find the largest power of 2 that can divide 268!

Solution $\left\lfloor \frac{268}{2} \right\rfloor + \left\lfloor \frac{268}{2^2} \right\rfloor + \left\lfloor \frac{268}{2^3} \right\rfloor + \left\lfloor \frac{268}{2^4} \right\rfloor$

$$+ \left\lfloor \frac{268}{2^5} \right\rfloor + \left\lfloor \frac{268}{2^6} \right\rfloor + \left\lfloor \frac{268}{2^7} \right\rfloor + \left\lfloor \frac{268}{2^8} \right\rfloor$$

$$= 134 + 67 + 33 + 16 + 8 + 4 + 2 + 1 = 265$$

Thus the greatest power of 2 is 265 that can divide exactly 268!

Dear students you might have noticed that if we further increase the value of $n!$ and consider a slightly higher prime. Number say 7,13, 17, 19 etc. for the divisibility, then we feel the real pain in solving the problem, since we don't know the values of higher powers of these divisors say 17^4, 17^5, 19^8, 19^9, ... etc. So this formula in this present form is not appropriate.

Now you must have noticed that we consider only integral values of the quotient q as $[q]$ which gives an integer just less than or equal to q. So we can do this calculation slightly in a easier manner.

Here we just divide the given number and then the succeeding quotients will be divided by the same remainder as in the case of successive division. So, for better understanding we can solve the previous problem in this manner as follows.

$$\begin{array}{r|l} 2 & 268 \\ \hline 2 & 134 \rightarrow \\ \hline 2 & 67 \rightarrow \\ \hline 2 & 33 \rightarrow \\ \hline 2 & 16 \rightarrow \\ \hline 2 & 8 \rightarrow \\ \hline 2 & 4 \rightarrow \\ \hline 2 & 2 \rightarrow \\ \hline 2 & 1 \rightarrow \\ \hline & 0 \end{array} \quad \text{265 (Add up all the quotients)}$$

[Divide successive quotients till you get 0 as the last quotient]

So we can frankly say that this method of calculation is easy since we need not to know the values of $2^1, 2^2, 2^3, 2^4, \ldots, 2^8$ etc. Also the division by 2 is easier than the division by 2^x, where x is any large integer.

Exp. 3) Find the largest power of 7 that can exactly divide 777!

Solution

$$\begin{array}{r|l} 7 & 777 \\ \hline 7 & 111 \rightarrow \\ \hline 7 & 15 \rightarrow \\ \hline 7 & 2 \rightarrow \\ \hline & 0 \end{array} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} 128$$

Thus the highest power of 7 is 128 by which 777! can be completely divided.

Exp. 4) Find the largest value of n in the 10^n which can exactly divide 1000!

Solution Since 10 is made up of 5 and 2 *i.e.*, $10 = 2 \times 5$.

So you can see that before 10! there is the presence of 10 in 5!, 6!, 7!, 8! or 9!.

So $\quad 10^n = (5 \times 2)^n = 5^n \times 2^n$

Thus we find the powers of 2 and 5 individually

$$\begin{array}{r|l} 2 & 1000 \\ \hline 2 & 500 \rightarrow \\ \hline 2 & 250 \rightarrow \\ \hline 2 & 125 \rightarrow \\ \hline 2 & 62 \rightarrow \\ \hline 2 & 31 \rightarrow \\ \hline 2 & 15 \rightarrow \\ \hline 2 & 7 \rightarrow \\ \hline 2 & 3 \rightarrow \\ \hline & 1 \rightarrow \end{array} \; 994 \quad \text{and} \quad \begin{array}{r|l} 5 & 1000 \\ \hline 5 & 200 \rightarrow \\ \hline 5 & 40 \rightarrow \\ \hline 5 & 8 \rightarrow \\ \hline & 1 \rightarrow \end{array} \; 249$$

So there is 2^{994} and 5^{249} but we can make only 249 combinations due to the restriction imposed by the lower power of 5!.

Thus the highest power of 10 is 249 which can divide 1000!

$$\left(\begin{array}{l} \because 2^{994} \times 5^{249} \Rightarrow 2^{745} \times 2^{249} \times 5^{249} \\ \Rightarrow 2^{745} \times (10)^{249} \end{array}\right)$$

Exp. 5) Find the number of zeros at the end of 1000!

Solution Since we know that the zeros at the end of any product are due to the presence of 10 as the factor of the product and the number of zeros depends upon the number of times 10 is involved. For example if there are seven 10s (*i.e.*, 7 combinations of 5×2) in the $n!$, then the number of zeros at the end of $n!$ will be 7.

But, since we have solved the same problem previous to this problem so we can conclude that the number of 10s in 1000! is 249. Hence the number of zeros at the end of the 1000! is 249.

Exp. 6) The number of zeros at the end of 100! is :

(a) 25 (b) 50
(c) 24 (d) 100

Solution

$$\begin{array}{r|l} 2 & 100 \\ \hline 2 & 50 \rightarrow \\ \hline 2 & 25 \rightarrow \\ \hline 2 & 12 \rightarrow \\ \hline 2 & 6 \rightarrow \\ \hline 2 & 3 \rightarrow \\ \hline & 1 \rightarrow \end{array} \; 97$$

and

$$\begin{array}{r|l} 5 & 100 \\ \hline 5 & 20 \rightarrow \\ \hline & 4 \rightarrow \end{array} \; 24$$

So, there will be only 24 combinations of 5×2, it means there will be 24 zeros at the end of 100!.

Hence (c) is the correct answer.

Exp. 7) Find the highest power of 63 which can completely divide 6336!.

Solution $\quad 63 = 3^2 \times 7$

Since this technique is applicable only for the prime factors. So we solve it by breaking 63 in its prime factors.

$$\begin{array}{r|l} 3 & 6336 \\ \hline 3 & 2112 \rightarrow \\ \hline 3 & 704 \rightarrow \\ \hline 3 & 234 \rightarrow \\ \hline 3 & 78 \rightarrow \\ \hline 3 & 26 \rightarrow \\ \hline 3 & 8 \rightarrow \\ \hline & 2 \rightarrow \end{array} \; 3164$$

and

$$\begin{array}{r|l} 7 & 6336 \\ \hline 7 & 905 \rightarrow \\ \hline 7 & 129 \rightarrow \\ \hline 7 & 18 \rightarrow \\ \hline & 2 \rightarrow \end{array} \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} 1054$$

$\because$ $63 = 9 \times 7 = 3^2 \times 7$

$\therefore$ Since to make a 9 we need 3×3 *i.e,* two times. So we divide 3164 by 2 and get 1582.

So $3^{3164} \times 7^{1054}$

$\Rightarrow$ $9^{1582} \times 7^{1054}$

Thus the largest power of 63 is 1054. That can completely divide 6336!

Exp. 8) Find the highest power of 40 which can completely divide 4000!

Solution $40 = 8 \times 5 = 2^3 \times 5$

So,

$$\begin{array}{r|l} 2 & 4000 \\ \hline 2 & 2000 \rightarrow \\ \hline 2 & 1000 \rightarrow \\ \hline 2 & 500 \rightarrow \\ \hline 2 & 250 \rightarrow \\ \hline 2 & 125 \rightarrow \\ \hline 2 & 62 \rightarrow \\ \hline 2 & 31 \rightarrow \\ \hline 2 & 15 \rightarrow \\ \hline 2 & 7 \rightarrow \\ \hline 2 & 3 \rightarrow \\ \hline & 1 \rightarrow \end{array} \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} 3994$$

and

$$\begin{array}{r|l} 5 & 4000 \\ \hline 5 & 800 \rightarrow \\ \hline 5 & 160 \rightarrow \\ \hline 5 & 32 \rightarrow \\ \hline 5 & 6 \rightarrow \\ \hline & 1 \rightarrow \end{array} \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} 999$$

Now since $2^{3m} \times 5^n = 8^m \times 5^n$

$\therefore$ $2^{3994} \times 5^{999} = (2^3)^{1331} \times 2 \times 5^{999}$

$\Rightarrow$ $2 \times (8^{1331} \times 5^{999})$

$\Rightarrow$ $2 \times 8^{332} \times (8 \times 5)^{999}$

Thus the highest power of 40 is 999 that can completely divide 4000!.

Exp. 9) Find the highest power of 81 that can completely divide 1800!.

Solution $81 = 3^4$

So,

$$\begin{array}{r|l} 3 & 1800 \\ \hline 3 & 600 \rightarrow \\ \hline 3 & 200 \rightarrow \\ \hline 3 & 66 \rightarrow \\ \hline 3 & 22 \rightarrow \\ \hline 3 & 7 \rightarrow \\ \hline & 2 \rightarrow \end{array} \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} 897$$

Thus we get 3^{897}. But we need 81.

Now since $81 = 3^4$

Therefore $3^{897} = (3^4)^{224} \times 3^1 = (81)^{224} \times 3$.

Therefore the highest power of 81 is 224 which can divide completely 1800! so $(81)^{224}$ can divide 1800!

NOTE Sometimes for your convenience you can skip some unnecessary calculation. As if you are asked to find the highest power of 30 which can completely divide 1357!

Since $30 = 2 \times 3 \times 5$. Now if you have been serious while doing the previous examples so you can conclude that you need not to solve to find the highest powers of 2 and 3. What you need is just to find the highest power of 5 (which is the greatest factor therefore it occurs least frequently) and as you know that the least frequent factor impose the restriction and hence it is the only effective value. Hence we just calculate the highest power of 5 and get the required result. Still there are some special points which are really difficult to explain on the paper.

These points can be effectively explained only in the classroom by teacher and last but not the least if you are intelligent enough so you can pick up these subtle but the important points by yourself as you are learning the concept religiously. Please note that there can be some shortcuts which I have not mentioned because it is difficult to explain exactly on paper and even some students don't fathom the logic of the shortcut. So it becomes dangerous and hence it is better to find the shortcuts by themselves just by intensive practice or seeking the help of an expert.

Concept of Unit Digit

Look at the following :

$$\begin{array}{r} 1 \times 5 = 5 \\ 3 \times 5 = 15 \\ 5 \times 5 = 25 \\ 7 \times 5 = 35 \\ 9 \times 5 = 45 \\ 11 \times 5 = 55 \\ \ldots \ \ldots \ \ldots \ \ldots \\ \ldots \ \ldots \ \ldots \ \ldots \end{array}$$

i.e., if the number whose last digit is 5, is multiplied by any odd number, the unit digit of the product will always be 5.

For example

$$13\times15=195, 19\times35=665 \text{ etc.}$$

Now,
$$2\times5=10$$
$$4\times5=20$$
$$6\times5=30$$
$$8\times5=40$$
$$10\times5=50$$
$$12\times5=60$$
$$\ldots\ \ldots\ \ldots\ \ldots$$
$$\ldots\ \ldots\ \ldots\ \ldots$$

i.e., if the number whose last is 5, is multiplied by any even number (including zero), the unit digit of the product is always zero. For example

$$82\times15=1230,$$
and $$156\times45=7020$$
and $$62\times13\times65=52390 \text{ etc.}$$

Now we will discuss that the unit digit of the resultant value depends upon the unit digits of all the participating numbers *i.e.,*

$$12+17+13+47=89$$

Thus it is clear that the unit digit of the resultant value 89 depends upon the unit digits 2, 7, 3, 7.

Similarly, $$6\times7\times9=378$$
$$3\times7\times8=168$$

So we can find out the unit digit of the resultant value only by solving the unit digits of the given expression.

Exp. 1) Find the unit digit of
$$123+345+780+65+44.$$

Solution We can find the unit digit just by adding the unit digits 3, 5, 0, 5, 4 as

$$3+5+0+5+4=17$$

So the unit digit (or the last digit) of the resultant value of the expression $123+345+780+65+44$ will be 7. (you can verify it by doing the whole sum)

Exp. 2) Find the unit digit of $676\times543\times19$.

Solution We can find the unit digit of the product of the given expression just by multiplying the unit digits (6, 3, 9) instead of doing the whole sum.

Thus $$6\times3\times9=162$$

Hence, the unit digit of the product of the given expression will be 2. (you can verify it by doing the complete sum)

Exp. 3) Find the unit digit of $135\times361\times970$.

Solution The unit digit can be obtained by multiplying the unit digits 5, 1, 0. then $5\times1\times0=0$ thus the unit digit will be zero.

Exp. 4) Find the unit digit of the product of all the odd prime numbers.

Solution The odd prime numbers are 3, 5, 7, 11, 13, 17, 19...etc.

Now we know that if 5 is multiplied by any odd number it always gives the last digit 5. So the required unit digit will be 5.

Cyclicity of the Unit Digit

Now, look at the following

$$1\times1=1$$
$$1\times1\times1=1$$
$$1\times1\times1\times1=1$$
$$1\times1\times1\times1\times1=1 \text{ etc.}$$

$$2^1=2$$
$$2^2=2\times2=4$$
$$2^3=2\times2\times2=8$$
$$2^4=2\times2\times2\times2=16$$
$$2^5=2\times2\times2\times2\times2=32$$
$$2^6=2\times2\times2\times2\times2\times2=64$$
$$2^7=2\times2\times2\times2\times2\times2\times2=128$$
$$2^8=2\times2\times2\times2\times2\times2\times2\times2=256$$
$$2^9=2\times2\times2\times2\times2\times2\times2\times2\times2=512$$
$$2^{10}=2\times2\times2\times2\times2\times2\times2\times2\times2\times2=1024$$
$$2^{11}=2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2=2048$$
$$2^{12}=2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2=4096$$

Similarly
$$3^1=3$$
$$3^2=9$$
$$3^3=27$$
$$3^4=81$$
$$3^5=243$$
$$3^6=729$$
$$3^7=2187$$
$$3^8=6561$$
$$3^9=19683 \text{ etc.}$$

Similarly,
$$4^1=4$$
$$4^2=16$$
$$4^3=64$$
$$4^4=256$$
$$4^5=1024$$
$$4^6=4096$$

Thus we can say that the unit digit follows a periodic pattern that is after a particular period it repeats in a cyclic form.

The unit digit of $2^1, 2^5, 2^9, 2^{13}, \ldots$ is the same which is 2.

Similarly $2^2, 2^6, 2^{10}, 2^{14}, \ldots$ etc. has the same unit digit, which is 4.

Again the last digit of 3^1 is 3.

and the last digit of 3^2 is 9.

and the last digit of 3^3 is 7.

and the last digit of 3^4 is 1.

and the last digit of 3^5 is 3.

and the last digit of 3^6 is 9.

and the last digit of 3^7 is 7.

and the last digit of 3^8 is 1.

Thus the last digit must follow a pattern. It can be seen that the last digit of 2 repeats after every four steps and the last digit of 4 repeats after every 2 steps.

NOTE

1. The last digit (or unit digit) of 0, 1, 5 and 6 is always the same irrespective of their powers raised to them.
2. The last digit of 4 and 9 follows the pattern of odd-even *i.e.,* their period is 2.
3. The last digit of 2, 3, 7, 8 repeats after every 4 steps *i.e.,* their cyclic period is 4.

Exp. 1) Find the last digit of 2^{35}.

Solution The last digit of $2^1 \Rightarrow 2$

$2^2 \Rightarrow 4$

$2^3 \Rightarrow 8$

$2^4 \Rightarrow 6$

$2^5 \Rightarrow 2$

Since its cyclic period is four , it means the unit digit of 2 will repeat after every 4 steps. Hence we can say that the last digit of $2^1, 2^5, 2^9, 2^{13}, 2^{17}, 2^{21}, 2^{25}, 2^{29}, 2^{33}, 2^{37}$ is the same and the last digit of $2^2, 2^6, 2^{10}, 2^{14}, 2^{18}, \ldots, 2^{30}, 2^{34}, 2^{38}$ is same and the last digit of $2^3, 2^7, 2^{11}, \ldots, 2^{31}, 2^{35}, 2^{39}$ is same and the last digit of $2^4, 2^8, 2^{12}, 2^{16}, \ldots, 2^{32}, 2^{36}, 2^{40}$ etc. is same.

Hence the last digit of 2^{35} is 8.

Alternatively You can see that the powers which are divisible by 4 (*i.e.,* cyclic period) give the same unit digit as 2^4. As $2^8, 2^{12}, 2^{16}, \ldots$ etc.

Again the powers which leave the remainder 1 when divided by 4 (which is the cyclic period) give the same unit digit as 2^1. As $2^5, 2^9, 2^{13}, \ldots$ etc.

Similarly the powers which leave the remainder 2 when divided by 4, give the same unit digit as 2^2. As $2^6, 2^{10}, 2^{14}, 2^{18}, \ldots$ etc.

Similarly the powers which leave the remainder 3 when divided by 4 give the same unit digit as 2^3. As $2^7, 2^{11}, 2^{15}, \ldots$ etc.

Since when 35 is divided by 4, the remainder is 3. Hence, the unit digit of 2^{35} is equal to the unit digit of 2^3, which is equal to 8.

Exp. 2) Find the unit digit of $(12)^{78}$.

Solution The unit digit of $(12)^{78}$ will be same as $(2)^{78}$. Now since we know that the cyclic period of unit digit of 2 is 4. The remainder when 78 is divided by 4 is 2. Hence the unit digit of 2^{78} will be same as 2^2 which is 4. Thus the unit digit of 12^{78} is 4.

Exp. 3) Find the unit digit of $(33)^{123}$.

Solution Since we know that the unit digit of $(33)^{123}$ will be same as $(3)^{123}$. Now the unit-digit of 3^{123} will be 7 since it will be equal to the unit digit of 3^3.

Thus the unit digit of $(3)^{123}$ is 7.

Exp. 4) Find the unit digit of $3^{47} + 7^{52}$.

Solution The unit digit of the given expression will be equal to the unit digit of the sum of the unit digits of both the terms individually.

Now, unit digit of 3^{47} is 7 (since it will be equal to 3^3) and the unit digit of 7^{52} is 1 (since it will be equal to 7^4)

Thus the unit digit of $3^{47} + 7^{52}$ is $7 + 1 = 8$.

Exp. 5) Find the unit digit of

$$3^6 \times 4^7 \times 6^3 \times 7^4 \times 8^2 \times 9^5.$$

Solution The unit digit of 3^6 is 9

The unit digit of 4^7 is 4

The unit digit of 6^3 is 6

The unit digit of 7^4 is 1

The unit digit of 8^2 is 4

The unit digit of 9^5 is 9

Therefore the unit digit of the given expression is 6, (since $9 \times 4 \times 6 \times 1 \times 4 \times 9 = 7776$).

Exp. 6) Find the unit digit of 111! (factorial 111).

Solution $111! = 1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times 110 \times 111$

Since there is a product of 5 and 2 hence it will give zero as the unit digit.

Hence the unit digit of 111! is 0 (zero).

Exp. 7) Find the unit digit of the product of all the prime number between 1 and $(11)^{11}$.

Solution The set of prime number $S = \{2, 3, 5, 7, 11, 13, \ldots\}$

Since there is one 5 and one 2 which gives 10 after multiplying mutually, it means the unit digit will be zero.

Exp. 8) Find the unit digit of the product of all the elements of the set which consists all the prime numbers greater than 2 but less than 222.

Solution The set of required prime numbers $= \{3, 5, 7, 11, \ldots\}$

Since there is no any even number in the set so when 5 is multiplied with any odd number, it always gives 5 as the last digit.

Hence the unit digit will be 5.

Exp. 9) Find the last digit of $222^{888} + 888^{222}$.

Solution The last digit of the expression will be same as the last digit of $2^{888} + 8^{222}$.

Now the last digit of 2^{888} is 6 and the last digit of the 8^{222} is 4. Thus the last digit of $2^{888} + 8^{222}$ is 0 (zero), since $6 + 4 = 10$.

Exp. 10) Find the last digit of $32^{32^{32}}$.

Solution The last digit of $32^{32^{32}}$ is same as $2^{32^{32}}$.

But $\quad 2^{32^{32}} = 2^{32 \times 32 \times 32 \times \ldots \times 32 \text{ times}}$

$\Rightarrow \quad 2^{32^{32}} = 2^{4 \times 8 \times (32 \times 32 \times \ldots \times 31 \text{ times})}$

$\Rightarrow \quad 2^{32^{32}} = 2^{4n}$,

where $n = 8 \times (32 \times 32 \times \ldots \times 31 \text{ times})$

Again $2^{4n} = (16)^n \Rightarrow$ unit digit is 6, for every $n \in N$

Hence, the required unit digit = 6.

Exp. 11) Find the last digit of the expression :

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots. + 100^2.$$

Solution The unit digit of the whole expression will be equal to the unit digit of the sum of the unit digits of the expression.

Now adding the unit digits of $1^2 + 2^2 + 3^2 + \ldots + 10^2$ we get

$$1 + 4 + 9 + 6 + 5 + 6 + 9 + 4 + 1 + 0 = 45$$

Hence, the unit digit of $1^2 + 2^2 + 2^2 + \ldots + 10^2$ is 5.

Now since there are 10 similar columns of numbers which will yield the same unit digit 5. Hence the sum of unit digits of all the 10 columns is 50 $(= 5 + 5 + 5 + \ldots + 5)$.

Hence, the unit digit of the given expression is 0 (zero).

Exp. 12) Find the unit digit of $1^1 + 2^2 + 3^3 + \ldots 10^{10}$.

Solution The unit digit of $1^1 = 1$

The unit digit of $2^2 = 4$

The unit digit of $3^3 = 7$

The unit digit of $4^4 = 6$

The unit digit of $5^5 = 5$

The unit digit of $6^6 = 6$

The unit digit of $7^7 = 3$

The unit digit of $8^8 = 6$

The unit digit of $9^9 = 9$

The unit digit of $10^{10} = 0$

Thus the unit digit of the given expression will be 7. $(\because 1 + 4 + 7 + 6 + 5 + 6 + 3 + 6 + 9 = 47)$

Exp. 13) Find the unit digit of

$$13^{24} \times 68^{57} + 24^{13} \times 57^{68} + 1234 + 5678.$$

Solution The unit digit of 3^{24} is 1

The unit digit of 8^{57} is 8

The unit digit of 4^{13} is 4

The unit digit of 7^{68} is 1

Therefore

$$= 1 \times 8 + 4 \times 1 + 4 + 8$$
$$= 8 + 4 + 4 + 8 = 24$$

Thus the unit digit of the whole expression is 4.

Exp. 14) The unit digit of the expression $\dfrac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}{100}$ is :

(a) 7 (b) 9
(c) 8 (d) none of these

Solution Since in the numerator of the product of the expression there will be 2 zeros at the end and these two zeros will be cancelled by 2 zeros of the denominator. Hence finally we get a non-zero unit digit in the expression.

Now, $\dfrac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}{100}$

$$= \frac{1 \times 2^8 \times 3^4 \times 5^2 \times 7^1}{5^2 \times 2^2}$$

$$= 1 \times 2^6 \times 3^4 \times 7$$

Therefore, the unit digit of the given expression will be same as that of $1 \times 2^6 \times 3^4 \times 7$.

Now, the unit digit of $1 \times 2^6 \times 3^4 \times 7$ is 8.

($\because$ the product of unit digits of $1, 2^6, 3^4, 7$ is $1 \times 4 \times 1 \times 7 = 28$)

Hence, the unit digit of $\dfrac{10!}{100}$ is 8.

Exp. 15) Find the unit digit of the expression

$$888^{9235!} + 222^{9235!} + 666^{2359!} + 999^{9999!}.$$

Solution First of all we individually need to find the unit digit individually of all the four terms

So, the unit digit of $888^{9235!}$ is equal to the unit digit of $8^{9235!}$

Now, the unit digit of $8^{9235!}$ is equal to the unit of 8^4 (since 9235! is divisible by 4), which is 6.

Again the unit digit of is $222^{9235!}$ is equal to the unit digit of $2^{9235!}$ and the unit digit of $2^{9235!}$ is same as that of 2^4 which is 6, since 9235! is divisible by 4. Further the unit digit of $666^{2359!}$ is 6, which is always constant for all the powers except zero and the unit digit of $999^{9999!}$ is 1 since the value of 9999! is even.

Thus the unit digit of the expression is 9. $(\because 6 + 6 + 6 + 1 = 19)$

Exp. 16) The last digit of the following expression is :

$$(1!)^1 + (2!)^2 + (3!)^3 + (4!)^4 + \ldots + (10!)^{10}$$

(a) 4 (b) 5
(c) 6 (d) 7

Solution The unit digit of the given expression will be equal to the unit digit of the sum of the unit digits of every term of the expression.

Now, The unit digit of $(1!)^2 = 1$

The unit digit of $(2!)^2 = 4$

The unit digit of $(3!)^3 = 6$

The unit digit of $(4!)^4 = 6$

The unit digit of $(5!)^5 = 0$

The unit digit of $(6!)^6 = 0$

Thus the unit digit of the $(7!)^7$, $(8!)^8$, $(9!)^9$, $(10!)^{10}$ will be zero.

So, the unit digit of the given expression $= 7$

$(\because 1 + 4 + 6 + 6 + 0 + 0 + 0 + 0 + 0 + 0 = 17)$

Exp. 17) The last 5 digits of the following expression will be $(1!)^5 + (2!)^4 + (3!)^3 + (4!)^2 + (5!)^1 + (10!)^5 + (100!)^4 + (1000!)^3 + (10000!)^2 + (100000!)$

(a) 45939 (b) 00929
(c) 20929 (d) can't be determined

Solution The last digit of $(1!)^5 = 1$

The last digit of $(2!)^4 = 16$

The last digit of $(3!)^3 = 216$

The last digit of $(4!)^2 = 576$

The last digit of $(5!)^1 = 120$

The last 5 digit of $(10!)^5 = 00000$

The last 5 digit of $(100!)^4 = 00000$

The last 5 digit of $(1000!)^3 = 00000$

The last 5 digit of $(10000!)^2 = 00000$

The last 5 digit of $(100000!)^1 = 00000$

Thus the last 5 digits of the given expression = 00929

$[\because 1 + 16 + 216 + 576 + 120 + 00000 + 00000 + 00000 + 00000 + 00000 = 00929]$

Involution

As in the previous articles we have studied that $a \,.\, a \,.\, a \ldots$ (n times) $= a^n$. So the process of multiplication of a number several times by itself is known as INVOLUTION.

Similarly, we use the same method in some algebraic expressions as follows :

1. $(a+b)^2 = a^2 + b^2 + 2ab = (a+b)(a+b)$
2. $(a-b)^2 = a^2 + b^2 - 2ab = (a-b)(a-b)$
3. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
   $= (a+b+c)(a+b+c)$
4. $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
5. $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$
6. $(a+b+c)^3 = a^3 + b^3 + c^3 + 3(a+b)(b+c)(c+a)$
7. $(a+b)^2 = (a-b)^2 + 4ab$
8. $(a-b)^2 = (a+b)^2 - 4ab$
9. $a^2 - b^2 = (a+b)(a-b)$
10. $a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$
11. $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$
12. $a^3 + b^3 + c^3 - 3abc = (a+b+c)$
    $(a^2 + b^2 + c^2 - ab - bc - ac)$

The above rules are used widely. But these are important in finding the square, cube, etc. of a number besides helping as find digits the remainder or divisibility of a number.

1. $a^n + b^n$ is divisible by $(a+b)$, when n is odd.
2. $a^n + b^n$ is never divisible by $(a-b)$
3. $a^n + b^n$ is not divisible by $(a+b)$, when n is even.
4. $a^n - b^n$ is always divisible by $(a-b)$
5. $a^n - b^n$ is divisible by $(a+b)$ only, when n is even.
6. $a^n - b^n$ is not divisible by $(a+b)$, when n is odd.

Exp. 1) Find the value of $107 \times 107 - 93 \times 93$

Solution $(107)^2 - (93)^2$ $[\because a^2 - b^2 = (a+b)(a-b)]$

$= (107 + 93)(107 - 93) = 200 \times 14 = 2800$

Exp. 2) Find the value of 734856×9999.

Solution $734856 \times 9999 = 734856 \times (10000 - 1)$

$= 7348560000 - 734856 = 7347825144$

Exp. 3) $6798 \times 223 + 6798 \times 77 = k$, then the value of k will be

(a) 2034900 (b) 3029400 (c) 2039400 (d) none

Solution $6798 \times 223 + 6798 \times 77 = 6798(223 + 77)$

$= 6798(300) = 2039400$

Exp. 4) $123 \times 123 + 77 \times 77 + 2 \times 123 \times 77 = ?$

Solution $123 \times 123 + 77 \times 77 + 2 \times 123 \times 77$

$= (123)^2 + (77)^2 + 2 \times 123 \times 77$

$= (123 + 77)^2$ $[(a+b)^2 = a^2 + b^2 + 2ab]$

$= (200)^2 = 40000$

Exp. 5) $\dfrac{(941 + 149)^2 + (941 - 149)^2}{(941 \times 941 + 149 \times 149)} = ?$

Solution $\dfrac{(941 + 149)^2 + (941 - 149)^2}{(941 \times 941 + 149 \times 149)}$

$$\because \quad \frac{(a+b)^2 + (a-b)^2}{a^2 + b^2} = \frac{(a^2 + b^2 + 2ab) + (a^2 + b^2 - 2ab)}{(a^2 + b^2)}$$

$$= \frac{2(a^2 + b^2)}{(a^2 + b^2)} = 2$$

Hence, the value of the given expression is 2. So you need not calculate the given numbers in the expression, where $a = 941$, $b = 149$.

Exp. 6) $\dfrac{888 \times 888 \times 888 - 222 \times 222 \times 222}{888 \times 888 + 888 \times 222 + 222 \times 222} = ?$

Solution $\dfrac{888 \times 888 \times 888 - 222 \times 222 \times 222}{888 \times 888 + 888 \times 222 + 222 \times 222}$

$$= \frac{(888)^3 - (222)^3}{(888)^2 + 888 \times 222 + (222)^2}$$

$$= \frac{(888)^3 - (222)^3}{(888)^2 + 888 \times 222 + (222)^2}$$

$$= 888 - 222$$

$$\left[\because \frac{a^3 - b^3}{(a^2 + b^2 + ab)} = \frac{(a-b)(a^2 + b^2 + ab)}{(a^2 + b^2 + ab)} = (a - b)\right]$$

$$= 666$$

Exp. 7) If $(64)^2 - (36)^2 = 20k$ then the value of k is :

(a) 140 (b) 120
(c) 80 (d) none of these

Solution $(64)^2 - (36)^2 = 20k$

$\Rightarrow$ $(64 + 36)(64 - 36) = 20k$

$\Rightarrow$ $100 . 28 = 20k$

$\Rightarrow$ $20 \times 5 \times 28 = 20k$

$\Rightarrow$ $k = 28 \times 5 = 140$

Hence (a) is the correct answer.

Exp. 8) If $a + \dfrac{1}{a} = 3$, then the value of $a^2 + \dfrac{1}{a^2}$ is :

(a) 6 (b) 7
(c) 9 (d) can't be determined

Solution $\because$ $\left(a + \dfrac{1}{a}\right) = 3$

$\therefore$ $\left(a + \dfrac{1}{a}\right)^2 = a^2 + \dfrac{1}{a^2} + 2a \times \dfrac{1}{a} = 9$

$\Rightarrow$ $a^2 + \dfrac{1}{a^2} + 2 = 9 \Rightarrow a^2 + \dfrac{1}{a^2} = 7$

Hence (b) is the correct option.

Exp. 9) If $a + \dfrac{1}{a} = 3$, then the value of $a^3 + \dfrac{1}{a^3}$ is :

(a) 15 (b) 18
(c) 27 (d) none of these

Solution $\left(a + \dfrac{1}{a}\right)^3 = a^3 + \dfrac{1}{a^3} + 3a . \dfrac{1}{a}\left(a + \dfrac{1}{a}\right)$

$\Rightarrow$ $(3)^3 = a^3 + \dfrac{1}{a^3} + 3(3)$

$$\left[\begin{array}{l}\because \quad a \times \dfrac{1}{a} = 1 \\ \text{and } (a+b)^3 = a^3 + b^3 + 3ab(a+b)\end{array}\right]$$

$\Rightarrow$ $27 = a^3 + \dfrac{1}{a^3} + 9 \Rightarrow a^3 + \dfrac{1}{a^3} = 18$

Hence (b) is the correct option.

Exp. 10) What is the value of

$$\frac{2.75 \times 2.75 \times 2.75 - 2.25 \times 2.25 \times 2.25}{2.75 \times 2.75 + 2.75 \times 2.25 + 2.25 \times 2.25}?$$

Solution $\because$ $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$

$\therefore$ $\dfrac{a^3 - b^3}{(a^2 + b^2 + ab)} = \dfrac{(a-b)(a^2 + b^2 + ab)}{(a^2 + b^2 + ab)} = (a - b)$

$\therefore$ The value of the expression $= 2.75 - 2.25 = 0.50$

Exp. 11) If $(\sqrt{a} + \sqrt{b}) = 17$ and $(\sqrt{a} - \sqrt{b}) = 1$, then the value of $\sqrt{ab}$ is :

(a) 72 (b) 27
(c) 35 (d) none of these

Solution $(x + y)^2 - (x - y)^2 = (x^2 + y^2 + 2xy) - (x^2 + y^2 - 2xy)$

$\Rightarrow$ $(x + y)^2 - (x - y)^2 = 4xy$

$\Rightarrow$ $xy = \dfrac{(x + y)^2 - (x - y)^2}{4}$

Thus $\sqrt{ab} = \dfrac{(\sqrt{a} + \sqrt{b})^2 - (\sqrt{a} - \sqrt{b})^2}{4}$

$$= \frac{(17)^2 - (1)^2}{4} = \frac{288}{4} = 72$$

Hence (a) is the correct option.

Exp. 12) Find the value of $a^3 + b^3 + c^3 - 3abc$ if $a + b + c = 12$ and $ab + bc + ca = 47$.

Solution $\because$ $a + b + c = 12$

$\therefore$ $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ac) = 144$

$\Rightarrow$ $a^2 + b^2 + c^2 + 2 \times 47 = 144$

$\Rightarrow$ $a^2 + b^2 + c^2 = 50$

Now, since $a^3 + b^3 + c^3 - 3abc$

$$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$$

or $a^3 + b^3 + c^3 - 3abc = 12(50 - 47) = 12 \times 3 = 36$

Exp. 13) If $a + b + c = 0$, then the value of $a^3 + b^3 + c^3$ is :

(a) 0 (b) abc
(c) $3abc$ (d) none of these

Solution $\because a^3 + b^3 + c^3 - 3abc$

$$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$$

or $a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow$ $a^3 + b^3 + c^3 = 3abc$

Hence (c) is the correct option.

Exp. 14) If $x + y + z = 0$, then the value of

$\dfrac{x^2y^2 + y^2z^2 + z^2x^2}{x^4 + y^4 + z^4}$ is :

(a) 0 (b) 1/2 (c) 1 (d) 2

Solution $\because$ $(x + y + z) = 0$

$\Rightarrow$ $x^2 + y^2 + z^2 + 2(xy + yz + zx) = 0$

$$\Rightarrow \quad x^2 + y^2 + z^2 = -2(xy + yz + zx)$$
$$\Rightarrow \quad (x^2 + y^2 + z^2)^2 = 4(xy + yz + zx)^2$$
$$\Rightarrow \quad x^4 + y^4 + z^4 + 2(x^2y^2 + y^2z^2 + z^2x^2)$$
$$= 4[x^2y^2 + y^2z^2 + z^2x^2 + 2xyz(x + y + z)]$$
$$\therefore \quad x^4 + y^4 + z^4 = 2(x^2y^2 + y^2z^2 + z^2x^2) \quad [\because (x + y + z) = 0]$$
$$\therefore \quad \frac{x^2y^2 + y^2z^2 + z^2x^2}{x^4 + y^4 + z^4} = \frac{1}{2}$$

Hence (b) is the correct option.

Exp. 15) The value of $a^3 + b^3 + c^3 - 3abc$, when $a = 87$, $b = -126$ and $c = 39$ is :

(a) 0 (b) 1259
(c) – 48 (d) none of these

Solution $\because \quad a + b + c = 0$

$$\therefore \quad a^3 + b^3 + c^3 - 3abc = 0$$

Thus (a) is the correct option.

Exp. 16) The greatest divisor of

$(a - b)(a + b)(a^2 + b^2)(a^4 + b^4)(a^8 + b^8)(a^{16} + b^{16})$ is :

(a) $a^{16} - b^{16}$ (b) $a^{16} + b^{16}$ (c) $a^{31} - b^{31}$ (d) $a^{32} - b^{32}$

Solution $(a - b)(a + b)(a^2 + b^2)(a^4 + b^4)(a^8 + b^8)(a^{16} + b^{16})$

$$= a^{32} - b^{32}$$

Hence, the greatest divisor is $a^{32} - b^{32}$. Hence (d) is the correct option.

Exp. 17) In the above example which of the following can not divide the given expression :

(a) $a^4 - b^4$ (b) $a^{12} - b^{12}$
(c) $a^{16} - b^{16}$ (d) can't be determined

Solution Since $a^{12} - b^{12}$ is not the factor of the given expression. Hence, it can not divide the given expression. Thus (b) is the appropriate option.

Exp. 18) The expression $31^n + 17^n$ is divisible by, for every odd positive integer n:

(a) 48 (b) 14
(c) 16 (d) both (a) and (c)

Solution $a^n + b^n$ is divisible by $(a + b)$ for every odd n, $n \in N$

Hence $31^n + 17^n$ is divisible by $48 (= 31 + 17)$ and since 16 is the factor of 48. So the given expression is divisible by both (a) and (c). Hence (d) is the most appropriate answer.

Exp. 19) Which is not the factor of $4^{6n} - 6^{4n}$ for any positive integer n?

(a) 5 (b) 25
(c) 7 (d) none of these

Solution $\because 4^{6n} - 6^{4n} = (64)^{2n} - (36)^{2n} = (64^n + 36^n)(64^n - 36^n)$

For $n = 1, 3, 5, \ldots$ etc. $(64^n + 36^n)$ is divisible by 100 and all its factors. Also $(64^n - 36^n)$ is divisible by 28 and all its factors.

Again for $n = 2, 4, 6, \ldots$ etc. $(64^n - 36^n)$ is always divisible by 100 and all its factors. Also it is divisible by 28 and all its factors. Hence (d).

Exp. 20) $19^n - 1$ is :

(a) always divisible by 9 (b) always divisible by 20
(c) is never divisible by 19 (d) only (a) and (c) are true

Solution $19^n - 1$ is divisible by $18 (= 19 - 1)$ when n is even or odd. So (a) is correct.

$19^n - 1$ is divisible by 20 only when n is even so (b) is wrong.

$19^n - 1$ is never divisible by 19 which is correct. Thus (d) is the most appropriate answer.

Concepts of Remainder

Some Important Results

1. Let us assume that when $a_1, a_2, a_3, \ldots, a_n$ are individually divided by d, the respective remainders are $R_1, R_2, R_3, \ldots, R_n$. Now, if we divide $(a_1 + a_2 + a_3 + \ldots + a_n)$ by d, we get the same remainder as when we get by dividing $(R_1 + R_2 + R_3 + \ldots + R_n)$ by d.

Exp. 1) Find the remainder when $38 + 71 + 85$ is divided by 16.

Solution When we divide 38, 71 and 85 we obtain the respective remainders as 6, 7 and 5. Now we can obtain the required remainder by dividing the sum of remainders (*i.e.*, $6 + 7 + 5 = 18$) by 16, which is 2 instead of dividing the sum of actual given numbers (*i.e.*, $38 + 71 + 85 = 194$) by 16, which also gives us the same remainder 2.

Exp. 2) Find the remainder when $1661 + 1551 + 1441 + 1331 + 1221$ is divided by 20.

Solution The remainders when 1661, 1551, 1441, 1331, 1221 are divided by 20 are 1, 11, 1, 11, 1. So the required remainder can be obtained just by dividing $1 + 11 + 1 + 11 + 1 (= 25)$ by 20. Hence the required remainder is 5.

Exp. 3) What is the remainder when $678 + 687 + 6879 + 6890$ is divided by 17?

Solution The individual remainders when 678, 687, 6879, 6890 are divided by 17 are 15, 7, 11, 5, respectively. Hence the required remainder can be obtained by dividing $15 + 7 + 11 + 5 (= 38)$ by 17. Thus the required remainder is 4.

Exp. 4) What is the remainder when $(10 + 10^2 + 10^3 + 10^4 + 10^5)$ is divided by 6?

Solution The remainder when 10 is divided by 6 is 4

The remainder when 100 is divided by 6 is 4

The remainder when 1000 is divided by 6 is 4

The remainder when 10000 is divided by 6 is 4

The remainder when 100000 is divided by 6 is 4

So the required remainder can be obtained by dividing $4 + 4 + 4 + 4 + 4 (= 20)$ by 6. Thus the required remainder is 2.

2. When 'a' is divided by 'd' the remainder is R and when 'a_1' and 'a_2' are divided by 'd' the remainders are R_1 and R_2, then the remainder R will be equal to the difference of remainders R_1 and R_2 if $a_1 - a_2 = a$.

   For example the remainder when 63 is divided by 35 is 28 and this can be obtained by taking the 63 as difference of any two numbers say 100 and 37 then the remainders when 100 and 37 are divided by 35 are 30 and 2 then the required remainder is $28\,(=30-2)$ [*i.e.*, Remainder of $\frac{63}{35}$

$$= \text{Remainder of } \frac{100}{35} - \text{Remainder of } \frac{37}{35} = 30 - 2 = 28]$$

NOTE If the obtained remainder in a division has negative sign, then the actual remainder can be obtained by adding the divisor with initially obtained remainder containing negative sign.

For example, the remainder of $\frac{63}{35}$

$$= \text{Remainder of } \frac{120}{35} - \text{Remainder of } \frac{57}{35} = 15 - 22 = -7$$

So, the required remainder $= 35 - 7 = 28$

3. When $a_1, a_2, a_3, \ldots$ are divided by a divisor d the respective remainders obtained are $R_1, R_2, R_3, \ldots$, then the remainder when $(a_1 \times a_2 \times a_3 \times \ldots)$ is divided by 'd' can be obtained by dividing $(R_1 \times R_2 \times R_3 \times \ldots.)$ by d. For example when $365 \times 375 \times 389$ is divided by 35 the remainders can be obtained by dividing the product of the remainders which are obtained by dividing 365, 375, 389 individually. Now since the remainders are 15, 25, 4 so the required remainder = Remainder of $(15 \times 25 \times 4)$ when divided by 35.

$$\text{Thus the required remainder} = \frac{15 \times 25 \times 4}{35} = \frac{60 \times 25}{35}$$
$$= \frac{25 \times 25}{35} = \frac{625}{35} = 30$$

   Thus 30 is the remainder.

NOTE The required remainder can be obtained by considering the remainders as the dividends.

Thus the required remainder $= R_1 \times R_2 \times R_3 \times R_4$
$= R_5 \times R_3 \times R_4 = R_6 \times R_4 = R_7$

where R_7 is the required remainder and $R_1, R_2, \ldots, R_6$ are the remainders which are obtained by dividing the remainders perse.

Exp. 1) Find the remainder when 123×1234 is divided by 15.

Solution The remainder of $\frac{123 \times 1234}{15}$ = Remainder of $\frac{3 \times 4}{15}$

(3 and 4 are the remainders of 123 and 1234 when divided by 15)

$$= \text{Remainder of } \frac{12}{15} = 12$$

Exp. 2) Find the remainder when $1719 \times 1721 \times 1723 \times 1725 \times 1727$ is divided by 18.

Solution Since you can understand that applying the traditional method of getting the remainder is very difficult since first of all you have to take the product of all the numbers as given in the expression, then this a very large product that has to be divided by 18, which is very lengthy and tedious process.

Thus we find the remainder of the remainders.

Hence, the remainder of

$$\frac{1719 \times 1721 \times 1723 \times 1725 \times 1727}{18}$$
$$= \frac{9 \times 11 \times 13 \times 15 \times 17}{18}$$

(9, 11, 13, 15, 17 are the remainders when the given numbers are divided by 18.)

$$= \frac{99 \times 13 \times 15 \times 17}{18}$$
$$= \frac{9 \times 13 \times 255}{18} = \frac{117 \times 255}{18}$$
$$= \frac{9 \times 3}{18} = \frac{27}{18} = 9$$

Exp. 3) Find the remainder when 7^7 is divided by 4.

Solution
$$\frac{7^7}{4} = \frac{7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7}{4}$$
$$= \frac{3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3}{4} = \frac{9 \times 9 \times 9 \times 3}{4}$$
$$= \frac{1 \times 1 \times 1 \times 3}{4} = \frac{3}{4}$$

Thus the required remainder is 3.

Exp. 4) Find the remainder when 11^8 is divided by 7.

Solution
$$\frac{11^8}{7} = \frac{11 \times 11 \times 11 \times 11 \times 11 \times 11 \times 11 \times 11}{7}$$
$$= \frac{4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{7}$$
$$= \frac{16 \times 16 \times 16 \times 16}{7}$$
$$= \frac{2 \times 2 \times 2 \times 2}{7} = \frac{16}{7}$$

Thus the required remainder = 2

Now you can see that, if in the above problem it is to be asked that find the remainder when 11^{75} is divided by 7, then it would be very difficult to solve it within a short span of time because it would involve a very large calculation if done in the above shown manner.

So follow the steps in which I will show you the appropriate process, which is generally known as ''pattern method to find the remainder.''

The remainder when 11^1 is divided by 7 is 4
The remainder when 11^2 is divided by 7 is 2
The remainder when 11^3 is divided by 7 is 1
The remainder when 11^4 is divided by 7 is 4
The remainder when 11^5 is divided by 7 is 2
The remainder when 11^6 is divided by 7 is 1
The remainder when 11^7 is divided by 7 is 4

So you can see that you are getting a pattern, if you follow it you can get the remainder for higher powers of 11.

Now you are seeing that after every 3 steps the cycle of remainders is repeating its course as after every 7 days a particular day repeats itself .

Now if we assume that the month of March starts with Monday, then which day will fall on 24th March of the same year? So you will quickly respond that it would be Wednesday.

Let us see how?

M	T	W	Th	F	Sat	Sun
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

As we can see that after every 7 days Monday comes on 1st, 8th, 15th, 22nd etc. So what we do is just divide the given date by 7. If it is divisible by 7 then the last day of the week (*i.e.,* last day of the cycle) will be the required day. If it is not divisible by 7 then it must leaves some remainder then we need to calculate the day according to the remainder.

For instance in the above example when we divide 24 by 7, then we get the remainder 3.

So we calculate the 3rd day starting from the first day of the week (or cycle). So the required day will be Wednesday.

Now, If I ask that which day will fall on 7th, 14th, 21st or 28th of this month then your response will be Sunday since all the dates given above are divisible by 7 which is the period of the cycle.

If the dates were 1, 8, 15, ... etc then the required day would have been Monday since when we divide these given dates by 7 the remainder is 1 so the dates which are corresponding to 1 *i.e.,* 8, 15, 22, 29 etc. will fall on Monday.

So I hope this discussion will be very helpful to you while solving the concerned problems. Thus we can say that the remainder when 11^{75} is divided by 7 is 1, since 75 is divisible by 3.

Exp. 5) Find the remainder when 5^{123} is divided by 7.

Solution The remainder when 5^1 is divided by 7 is 5
The remainder when 5^2 is divided by 7 is 4
The remainder when 5^3 is divided by 7 is 6
The remainder when 5^4 is divided by 7 is 2
The remainder when 5^5 is divided by 7 is 3
The remainder when 5^6 is divided by 7 is 1
The remainder when 5^7 is divided by 7 is 5
The remainder when 5^8 is divided by 7 is 4 etc.

So we see that the cyclic period of remainder is 6,since after 6 steps the remainders start repeating.

Now we divide the power of 5 *i.e.,* 123 by 6, then it leaves the remainder 3.

It means the required remainder will be equal to the corresponding remainder when $5^3$is divided by 7, which is 6.

So the remainder when 5^{123} is divided by 7 is 6.

Explanation : Remainder $\frac{5^{123}}{7}$ = Remainder $\frac{5^{120}\times 5^3}{7}$

= Remainder $\frac{(5^6)^{20}\times 5^3}{7}$

= Remainder = $\frac{5^3}{7}$

= Remainder is 6.

NOTE Remember that when you get the remainder 1 while obtaining the cyclic period, you can stop the further calculation because at this step your cyclic period ends and in the next step the remainder starts repeating as in the above example at 5^6 we get the remainder 1.
So we need not calculate further for $5^7, 5^8, 5^9$ etc., because all these numbers will give the same remainders as $5^1, 5^2, 5^3, \ldots$, etc. respectively.

Exp. 6) Find the remainder when 123^{321} is divided by 5.

Solution The remainder when 123 is divided by 5 is 3

So, the remainder when 123^{321} is divided by 5 is same as when 3^{321} is divided by 5.

Now, the remainder when 3^1 is divided by 5 is 3
The remainder when 3^2 is divided by 5 is 4
The remainder when 3^3 is divided by 5 is 2
The remainder when 3^4 is divided by 5 is 1
The remainder when 3^5 is divided by 5 is 3

So the cyclic period is 4 since at 3^4 we get the remainder 1 (after which the cycle starts repeating).

Thus the remainder when 3^{321} is divided by 5 is 3 since we get the remainder 1 when 321 is divided by 4 (the cyclic period).

or Rem. $\frac{123^{321}}{5}$ = Rem. $\frac{3^{321}}{5}$ = Rem. $\frac{3^{320}\times 3^1}{5}$

= Rem . $\frac{(3^4)^{80}\times 3^1}{5}$ = Rem. $\frac{3^1}{5}$

= Remainder is 3.

Exp. 7) Find the remainder when $923^{888}+235^{222}$ is divided by 4.

Solution Rem $\frac{923^{888}+235^{222}}{4}$

= Rem. $\frac{3^{888}+3^{222}}{4}$

= Rem. $\frac{1+1}{4} = \frac{2}{4}$

Thus the remainder is 2.

Exp. 8) Find the remainder of $\dfrac{3^{9415}}{80}$.

Solution Rem $\dfrac{3^{9415}}{80}$ = Rem $\dfrac{3^{9412}\times 3^3}{80}$

= Rem $\dfrac{(3^4)^{2353}\times 3^3}{80}$ [Since the cyclic period is 4]

= Rem $\dfrac{(81)^{2353}\times 3^3}{80}$ = Rem $\dfrac{1\times 27}{80}$ = Rem $\dfrac{27}{80}$

Thus the remainder is 27.

Exp. 9) Find the remainder when
$10^1+10^2+10^3+10^4+10^5+\ldots+10^{99}$ is divided by 6.

Solution The remainder when 10^1 is divided by 6 is 4
The remainder when 10^2 is divided by 6 is 4
The remainder when 10^3 is divided by 6 is 4
The remainder when 10^4 is divided by 6 is 4
The remainder when 10^5 is divided by 6 is 4
Thus the remainder is always 4.

So, the required remainder = $\dfrac{4+4+4+\ldots 99 \text{ times}}{6} = \dfrac{396}{6}$

Thus the remainder is zero.

Exp. 10) The remainder of $\dfrac{888^{222}+222^{888}}{3}$ is :

(a) 0 (b) 1 (c) 2 (d) 3

Solution Remainder $\dfrac{888^{222}+222^{888}}{3}$ = Remainder is zero,

since 888 and 222 both (bases) are divisible by 3.

Exp. 11) The remainder of $\dfrac{888^{222}+222^{888}}{5}$ is :

(a) 0 (b) 1 (c) 3 (d) 4

Solution Rem. $\dfrac{888^{222}+222^{888}}{5}$

= Rem. $\dfrac{888^{222}}{5}$ + Rem. $\dfrac{222^{888}}{5}$

= Rem. $\dfrac{3^{222}}{5}$ + Rem. $\dfrac{2^{888}}{5}$

= Rem. $\dfrac{(3^4)^{55}\times 3^2}{5}$ + Rem. $\dfrac{(2^4)^{222}}{5}$

= Rem. $\dfrac{1\times 9}{5}$ + Rem. 1

= Rem. $\dfrac{4}{5}$ + Rem. $\dfrac{1}{5}$ = Rem. $\dfrac{4+1}{5} \Rightarrow \dfrac{5}{5}$

Thus the remainder is zero.

Alternatively [To check the divisibility by 5 just see the sum of the unit digits which is 10 (= 4 +6)

$\because$ $8^{222} \rightarrow 4$ (unit digit)

and $2^{888} \rightarrow 6$ (unit digit)

Hence it is divisible. So there is no remainder]

Exp. 12) The remainder of $\dfrac{32^{32^{32}}}{7}$:

(a) 2 (b) 1 (c) 4 (d) none

Solution $32^{32^{32}}$ means $32^{(32\,.\,32\,.\,32\,.\,32 \ldots 32 \text{ times})}$

Now the remainder of $\dfrac{32}{7}$ is 4.

Again $\dfrac{4^{32\,.\,32\,.\,32\,.\,32\,\ldots\,32 \text{ times}}}{7} \Rightarrow \dfrac{4^{2\,.\,2\,.\,2\,\ldots\,32 \text{ times}}}{7}$

$\left[\because \dfrac{4^1}{7}\rightarrow 4, \dfrac{4^2}{7}\rightarrow 2, \dfrac{4^3}{7}\rightarrow 1\right]$,

Remainder = 4

Since remainder of $\dfrac{4^2}{7} = \dfrac{4^8}{7} = \dfrac{4^{32}}{7} = \dfrac{4^{128}}{7} = \ldots$ is 2

and remainder of $\dfrac{4^4}{7} = \dfrac{4^{16}}{7} = \dfrac{4^{64}}{7} = \dfrac{4^{256}}{7} = \ldots$ is 4

NOTE 1. $\dfrac{(a+1)^n}{a}$ gives always remainder 1.

2. $\dfrac{a^n}{(a+1)}$ gives remainder 1 when n is even and gives the remainder a itself when n is odd, where a is any integer and n being the positive integer.

Exp. 13) The remainder when 8^{1785} is divided by 7 is :

(a) 5 (b) 1
(c) 6 (d) can't be determined

Solution $\because \dfrac{(a+1)^n}{a}$ leaves always remainder 1.

So $\dfrac{8^{1785}}{7} = \dfrac{(7+1)^{1785}}{7}$ gives the remainder 1.

Exp. 14) The remainder when $(16)^{3500}$ is divided by 17 is

(a) 1 (b) 0 (c) 16 (d) none

Solution $\because \dfrac{a^n}{a+1}$ gives remainder 1 when n is even

So the remainder of $\dfrac{16^{3500}}{17} = \dfrac{16^{3500}}{(16+1)}$ is 1.

Hence (a) is the correct option.

Exp. 15) The remainder when $(3)^{81}$ is divided by 28 is :

(a) 1 (b) 8 (c) 18 (d) 26

Solution The remainder of $\dfrac{3^{81}}{28}\left[=\dfrac{(3^3)^{27}}{28}=\dfrac{(27)^{27}}{28}\right]$ is 27.

Since $\dfrac{a^n}{(a+1)}$ gives remainder a when n is odd.

Exp. 16) The remainder of $\dfrac{2^{243}}{3^2}$ is :

(a) 8 (b) 10 (c) 4 (d) none

Solution $\dfrac{2^{243}}{3^2} = \dfrac{(2^3)^{81}}{9} = \dfrac{8^{81}}{9} = \dfrac{8^{81}}{(8+1)}$

Hence the remainder is 8 since the power of 8 is odd.

Exp. 17) The remainder when $(3)^{67!}$ is divided by 80 :

(a) 0 (b) 1
(c) 2 (d) can't be determined

Solution $\because 3^4 = 81$, so $\frac{3^{4n}}{80}$ gives the remainder 1.

Thus $\frac{3^{67!}}{80}$ will also give the remainder 1.

Since $67! = 4n$ for a positive integer n.

Exp. 18) The remainder of $\frac{39^{93!}}{40}$ is :

(a) 0 (b) 1 (c) 39 (d) 13

Solution $\because \frac{a^n}{a+1}$ gives remainder 1, when n is even, Now since 93! is an even number, hence the remainder is 1. Thus (b) is the correct choice.

Exp. 19) The remainder of $\frac{2^{59!}}{255}$ is :

(a) 0 (b) 1 (c) 55 (d) 56

Solution $2^{59!}$ can be expressed as 2^{8n}

So, $$\text{Rem. } \frac{2^{59!}}{255} = \text{Rem. } \frac{2^{8n}}{255} = \text{Rem. } \frac{(256)^n}{255}$$

Hence the remainder is 1.

Introductory Exercise 1.9

1. Find the last two digits of 64^{81}
(a) 84 (b) 24
(c) 64 (d) 44

2. If $x + \frac{1}{x} = 2$, then the value of $x^2 + \frac{1}{x^2}$ is :
(a) 6 (b) 4
(c) 2 (d) 0

3. If $a = 24, b = 26, c = 28$, then the value of $a^2 + b^2 + c^2 - ab - bc - ac$ will be :
(a) 0 (b) 4
(c) 8 (d) 12

4. If $(a^2 + b^2)^3 = (a^3 + b^3)^2$ and $ab \neq 0$ then $\left(\frac{a}{b} + \frac{b}{a}\right)^6$ is equal to :
(a) $\frac{a^6 + b^6}{a^3 b^3}$ (b) $\frac{64}{729}$
(c) 1 (d) $\frac{a^6 + a^3 b^3 + b^6}{a^2 b^4 + a^4 b^2}$

5. If $x - y = 1$, then the value of $x^3 - y^3 - 3xy$ will be :
(a) 1 (b) –1
(c) 3 (d) –3

6. If $x = 1234, y = 4321, z = -5555$, what is the value of $\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy}$ is:
(a) 0 (b) 1
(c) 3 (d) 9/4

7. Find the remainder when $2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^{255}$ is divided by 255.
(a) 0 (b) 1
(c) 127 (d) none of these

8. Among the expression $(1 - 3p)$, $[1 - (3p)^2]$, $[1 - (3p)^3]$ and $[1 - (3p)^4]$ the number of factors of $(1 - 81p^4)$ is :
(a) one (b) two (c) four (d) three

9. The remainder when $x^4 - y^4$ is divided by $x - y$ is :
(a) 0 (b) $x + y$
(c) $x^2 - y^2$ (d) $2y^4$

10. If $x - \frac{1}{x} = 2$, then the value of $x^4 + \frac{1}{x^4}$ is :
(a) 4 (b) 8
(c) 12 (d) 34

11. If the sum and the product of two numbers are 25 and 144; respectively, then the difference of the numbers must be :
(a) 3 (b) 5
(c) 7 (d) 11

12. The sum of two numbers is 9 and the sum of their squares is 41. The numbers are :
(a) 4 & 5 (b) 1 & 8
(c) 3 & 6 (d) 2 & 7

13. The square root of $\frac{\left(3\frac{1}{4}\right)^4 - \left(4\frac{1}{3}\right)^4}{\left(3\frac{1}{4}\right)^2 - \left(4\frac{1}{3}\right)^2}$:
(a) $5\frac{7}{12}$ (b) $5\frac{5}{12}$
(c) $5\frac{12}{13}$ (d) $5\frac{1}{6}$

14. The value of $\frac{(119)^2 + (119)(111) + (111)^2}{(119)^3 - (111)^3}$ is :
(a) 8 (b) $\frac{1}{8}$ (c) 230 (d) $\frac{1}{230}$

15. The value of
$$\frac{(1.5)^3 + (4.7)^3 + (3.8)^3 - 3 \times 1.5 \times 4.7 \times 3.8}{(1.5)^2 + (4.7)^2 + (3.8)^2 - (1.5 \times 4.7) - (4.7 \times 3.8) - (1.5 \times 3.8)}$$ is :
(a) 8 (b) 9
(c) 10 (d) 11

16. Simplified value of $\dfrac{8.73\times 8.73\times 8.73+4.27\times 4.27\times 4.27}{8.73\times 8.73-8.73\times 4.27+4.27\times 4.27}$ is :

(a) 11 (b) 12 (c) 13 (d) 14

17. The expression $(a-b)^3+(b-c)^3+(c-a)^3=0$ if :

(a) $a<b<c$ (b) $a>b>c$
(c) $a=b=c$ (d) $a\neq b=c$

18. If $a+b+c=11, a^2+b^2+c^2=51$, what is the value of $ab+bc+ac$?

(a) 24 (b) 28 (c) 32 (d) 35

19. If $x+y+z=0$, then the value of $\dfrac{x^2}{yz}+\dfrac{y^2}{zx}+\dfrac{z^2}{xy}$ is :

(a) 3 (b) −3 (c) 0 (d) $3xyz$

20. If $\left(x-\dfrac{1}{x}\right)=5$, then $x^3-\dfrac{1}{x^3}$ equals :

(a) 125 (b) 130
(c) 135 (d) 140

21. The value of $\left(\dfrac{e^x+e^{-x}}{2}\right)^2-\left(\dfrac{e^x-e^{-x}}{2}\right)^2$ is :

(a) 0 (b) 1 (c) 4 (d) 16

22. The continued product of $(1+x), (1+x^2), (1+x^4), (1+x^8)$ and $(1-x)$ is :

(a) $(1-x^8+x^{16})$
(b) (x^8+x^{16})
(c) $(1-x^{16})$
(d) $(x^{16}-1)$

1.10 Rational and Irrational Numbers

Rational Numbers

The numbers which can be expressed as the ratio of any two integers *i.e,* in the form of $\dfrac{p}{q}$, where p, q are integers, prime to each other (*i.e,* p and q must be coprime) and $q\neq 0$ are called the rational numbers. These numbers are denoted by 'Q'. Thus $3, -4, \dfrac{1}{7}, -\dfrac{3}{8}, \sqrt{16}, \sqrt{25}$ etc. are rational numbers. All integers and all fractions are rational numbers, which are also called commensurable quantities.

Facts About Rational Numbers

1. The denominator of any rational number can never be zero since the division by zero is undefined.
2. Every integer *e.g.,* $0, \pm 1, \pm 2, \pm 3..$ etc. is a rational number since it can be expressed as
$7=\dfrac{7}{1}, -3=-\dfrac{3}{1}, 0=\dfrac{0}{1}$ etc.
3. All the decimal numbers which are terminating are rational numbers *e.g.,*
$0.14==\dfrac{14}{100}, -8.5=\dfrac{-85}{10}, 0.006=\dfrac{6}{1000}$ etc.
4. All the recurring decimals, which are non terminating, are the rational numbers.
e.g., 0.4444..., 1.4141...., 6.785785..., 1.42857142857 ... $\dfrac{22}{7}$ etc.

NOTE π is not a rational number, since the exact value of π is not $\dfrac{22}{7}$.

Properties of Rational Numbers

1. **Closure law** Addition and multiplication of any two rational number is also a rational number.
2. **Commutative law** For any two rational numbers $\dfrac{a}{b}, \dfrac{c}{d}$. $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{c}{d}+\dfrac{a}{b}$ and $\dfrac{a}{b}\times\dfrac{c}{d}=\dfrac{c}{d}\times\dfrac{a}{b}$
3. **Associative law :** For any three rational numbers $\dfrac{a}{b}, \dfrac{c}{d}$ and $\dfrac{e}{f}$.
$$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$$
and $$\left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot\frac{e}{f}=\frac{a}{b}\cdot\left(\frac{c}{d}\cdot\frac{e}{f}\right)$$
4. **Additive and Multiplicative Identity :**
$\dfrac{a}{b}+0=\dfrac{a}{b}$ $\quad(\because 0=\dfrac{0}{1}$ is an additive identity$)$
and $\dfrac{a}{b}\times 1=\dfrac{a}{b}$ $\quad(\because 1=\dfrac{1}{1}$ is a multiplicative identity$)$
5. **Additive and Multiplicative Inverse :**
$\dfrac{a}{b}+\left(-\dfrac{a}{b}\right)=\dfrac{0}{1}$ $\quad(-\dfrac{a}{b}$ is additive inverse of $\dfrac{a}{b})$
$\dfrac{a}{b}\times\left(\dfrac{b}{a}\right)=\dfrac{1}{1}$ $\quad(\dfrac{b}{a}$ is multiplicative inverse of $\dfrac{a}{b})$
where $a\neq 0, b\neq 0$
6. **Rational Numbers** Do not follow the Commutative law and Associative law for the subtraction and division.

7. **Distribution law :** For any three rational numbers a, b, c $a.(b+c) = a.b + a.c$ and $a.(b-c) = a.b - a.c$
8. If x and y are any two rational numbers such that $x < y$ then $\frac{1}{2}(x+y)$ is also a rational number lying between x and y which shows that **there are infinite number of rational numbers between any two rational numbers**.

Irrational Numbers

The numbers which can't be expressed in the form $\frac{p}{q}$, where p, q are two integers prime to each other and $q \neq 0$ are called irrational numbers. Thus $\sqrt{2}, \sqrt{3}, -\sqrt{3}, \sqrt[3]{4}, \sqrt[3]{6}, \sqrt{5}, \pi$, etc. are irrational numbers. The decimal representation of these numbers is non-repeating and non-terminating.

e.g., 7.2030030003 ..., $\sqrt{2} = 1.41421....$, $\sqrt{3} = 1.732....$

Properties of Irrational Numbers

(i) The set 'P' of all irrational numbers is not closed for addition since the sum of two irrationals need not be irrational *e.g.,* $(3+\sqrt{5}) \in P$, $(3-\sqrt{5}) \in P$ but $(3+\sqrt{5}) + (3-\sqrt{5}) = 6 \notin P$

(ii) The set P of all irrational number is not closed for multiplication since the product of two irrationals need not be irrational.
e.g., $\sqrt{3} \in P, -\sqrt{3} \in P$ but $\sqrt{3} \times -\sqrt{3} = -3 \notin P$

Real Numbers

All the rational and all irrational numbers are called as real numbers *i.e,* the set of real numbers is the union of entire rational and irrational numbers.

e.g., $-3, 2, 0, \frac{5}{7}, \sqrt{4}, \sqrt{2}, \sqrt{3}, \pi, e, \frac{22}{7}, 0$, etc.

NOTE For some real numbers a and b

1. $a > b \Rightarrow -a < -b$
2. $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$
3. $a > b \Rightarrow a + k > b + k$
4. $a > b \Rightarrow ak > bk$, when $k > 0$
5. $a.b = 0 \Rightarrow$ either a is zero or b is zero or both a and b are zero.

Imaginary Numbers

If the square of a number is negative then this number is called as an imaginary number. *e.g.,* $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}$ etc.

An imaginary number is denoted by 'i', where $i = \sqrt{-1}$

Facts About Imaginary Numbers

1. $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$, etc.
2. $i^n = i^{4p+r} = (i^4)^p \times i^r = 1 \times i^r = i^r$
3. $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$, if both a, b are negative *i.e.,* imaginary otherwise at least one of a, b must be whole number *e.g.,*

$$\sqrt{-5} \times \sqrt{-7} = \sqrt{5}i \times \sqrt{7}i = \sqrt{35}i^2 = -\sqrt{35}$$

but $\sqrt{-5} \times \sqrt{-7} = \sqrt{-5 \times -7} \neq \sqrt{35}$

Exp. 1) Find the value of i^{63}.

Solution $i^{63} = (i^4)^{15} \times i^3 = i^3 = -i$

Exp. 2) Find the value of i^{170}.

Solution $i^{170} = (i^4)^{42} \times i^2 = i^2 = -1$

Exp. 3) Find the value of $\frac{1}{i^{123}}$

Solution $\frac{1}{i^{123}} = \frac{1}{i^{120} \times i^3} = \frac{1}{1 \times i^3} = \frac{1}{i^3} = \frac{1}{i^3} \times \frac{i}{i} = \frac{i}{i^4} = i$

Exp. 4) Find the value of $\sqrt{-16} \times \sqrt{-25}$.

Solution $\sqrt{-16} \times \sqrt{-25} = 4i \times 5i = 20i^2 = -20$

Exp. 5) Find the value of $(\sqrt{-1})^{4n+1}$.

Solution $(\sqrt{-1})^{4n+1} = i^{4n+1} = i^{4n} \times i = i$

Exp. 6) Find the value of i^{-343}.

Solution $i^{-343} = \frac{1}{i^{343}} = \frac{1}{(i^4)^{85} \times i^3} = \frac{1}{1 \times i^3} = \frac{1 \times i}{i^3 \times i} = \frac{i}{i^4} = i$

Exp. 7) Find the value of $i^{248} + i^{341} + i^{442} + i^{543}$.

Solution $i^{248} + i^{341} + i^{442} + i^{543} = 1 + i + (-1) + (-i) = 0$

Exp. 8) The value of $\frac{1}{i^n} + \frac{1}{i^{(n+3)}} + \frac{1}{i^{(n+2)}} + \frac{1}{i^{(n+1)}}$ is :

(a) -1 (b) 1
(c) 0 (d) can't be determined

Solution $\frac{1}{i^n} + \frac{1}{i^{(n+3)}} + \frac{1}{i^{(n+2)}} + \frac{1}{i^{(n+1)}}$ is :

$$= \frac{1}{i^n} + \frac{1}{i^n \times i^3} + \frac{1}{i^n \times i^2} + \frac{1}{i^n \times i}$$

$$= \frac{1}{i^n} + \frac{1}{i^n} \times \frac{i}{i^4} + \frac{1}{i^n} \times \frac{i^2}{i^4} + \frac{1}{i^n} \times \frac{i^3}{i^4}$$

$$= \frac{1}{i^n} + \frac{i}{i^n} + \frac{i^2}{i^n} + \frac{i^3}{i^n} = \frac{1}{i^n} + \frac{i}{i^n} + \frac{-1}{i^n} - \frac{i}{i^n} = 0$$

Hence (c) is the correct option.

Complex Numbers

The combination of real number and imaginary number is known as complex number *i.e.,* the number of the form $a + ib$ where a and b are purely real numbers and $i = \sqrt{-1}$, is called as a complex number. It is denoted by $z = a + ib$ where real $z = a$ and imaginary $z = b$.

1. **Property of order :** For any two complex numbers $(a+ib)$ and $(c+id)$ $(a+ib) < (\text{or} >)\ c+id$ is not defined *e.g.,* $9+4i < 3-i$ makes no sense.
2. A complex number is said to be purely real if $\text{Im}\,(z) = 0$ and it is said to be purely imaginary if $Re(z) = 0$. The complex number $0 = 0 + i.0$ is both purely real and purely imaginary.
3. The sum of four consecutive powers of i is zero.

   i.e., $i^k + i^{k+1} + i^{k+2} + i^{k+3} = 0$

Operations on Complex Numbers

Let z_1, z_2 be two complex numbers such that $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$ then :

1. $z_1 \pm z_2 = (a_1 \pm a_2) + i(b_1 \pm b_2)$
2. $z_1 . z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$
3. $z_1 = z_2 \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$

Conjugate Complex Number

The complex number $z = a + ib$ and $\bar{z} = (a - ib)$ are called the complex conjugate of each other, where $i = \sqrt{-1}$, $b \neq 0$ and a, b are real numbers.

Properties of Conjugate

For any z there exists the mirror image of z along the real axis denoted as $\bar{z}$, then

(i) $z = \bar{z} \Leftrightarrow z$ is purely real

(ii) $z = -\bar{z} \Leftrightarrow z$ is purely imaginary

(iii) $\overline{(\bar{z})} = z$

(iv) $\text{Re}\,(z) = \text{Re}\,(\bar{z}) = \dfrac{z+\bar{z}}{2}$

(v) $\text{Im}\,(z) = \dfrac{z-\bar{z}}{2i}$

(vi) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(vii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(viii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(ix) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

(x) $z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2\,\text{Re}\,(\bar{z}_1 z_2) = 2\,\text{Re}\,(z_1 \bar{z}_2)$

(xi) $\overline{z^n} = (\bar{z})^n$

(xii) If $z = f(z_1)$, then $\bar{z} = f(\bar{z}_1)$

Modulus of a Complex Number

The modulus of a complex number

$$z = (a + ib) \text{ is represented as } |z| = \sqrt{a^2 + b^2}.$$

Properties of Modulus

(i) $|z| \geq 0 \Rightarrow |z| = 0$ iff $z = 0$ and $|z| > 0$ iff $z \neq 0$

(ii) $-|z| \leq \text{Re}\,(z) \leq |z|$ and $-|z| \leq \text{Im}\,(z) \leq |z|$

(iii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iv) $z\bar{z} = |z|^2$

(v) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

(vi) $|z_1 \pm z_2 \pm z_3 \pm \ldots z_n|$
$$\leq |z_1| \pm |z_2| \pm |z_3| \pm \ldots |z_n|$$

(vii) $|z_1 \pm z_2| \geq ||z_1| - |z_2||$

(viii) $|z_1 z_2| = |z_1||z_2|$

(ix) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(x) $|z^n| = |z|^n$

(xi) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

(xii) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\bar{z}_1 \pm \bar{z}_2)$
$$= |z_1|^2 + |z_2|^2 \pm (z_1 \bar{z}_2 + \bar{z}_1 z_2)$$

(xiii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

nth root of Unity

1. Unity has n roots namely $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$ which are in geometric progression and the sum of these n roots is zero.
2. Product of n roots is $(-1)^{n-1}$.

Square Root

The square root of $z = a + ib$ are

$$\pm\left(\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right) \text{for } b > 0$$

and $$\pm\left(\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right) \text{for } b < 0$$

Remember

1. The square root of i are $\pm\left(\frac{1+i}{\sqrt{2}}\right)$ when $b=1$
2. The square root of $-i$ are $\left(\frac{1-i}{\sqrt{2}}\right)$ when $b=-1$

or $\sqrt{a+ib}=\pm(x+iy)$ and $\sqrt{a-ib}=\pm(x-iy)$

where $x=\left(\frac{a+\sqrt{a^2+b^2}}{2}\right)^{1/2}$ and $y=\left(\frac{\sqrt{a^2+b^2}-a}{2}\right)^{1/2}$

Cube root of unity

Cube root of unity are $1\,\omega,\omega^2$ where

(a) $1+\omega+\omega^2=0$ (b) $\omega^3=1$

(c) $\overline{\omega}=\omega^2$ and $(\overline{\omega})^2=\omega$

(d) $\omega^{3n}=1, \omega^{3n+1}=\omega, \omega^{3n+2}=\omega^2$

NOTE Iota (i) is neither 0, nor greater than 0, or nor less than 0.

Exp. 1) Find the modulus of $5+\sqrt{-11}$.

Solution $\because$ $z=5+\sqrt{-11}=5+\sqrt{11}\,i$

$\therefore$ $|z|=\sqrt{25+11}=\sqrt{36}=6$

Exp. 2) If the multiplicative inverse of a complex number is $\frac{\sqrt{5}+6i}{41}$ then the complex number itself is:

(a) $\sqrt{5}-6i$ (b) $\sqrt{5}+6i$

(c) $6+\sqrt{7}i$ (d) none of these

Solution Since $z\cdot\frac{(\sqrt{5}+6i)}{41}=1$

$\Rightarrow$ $z=\frac{41}{(\sqrt{5}+6i)}\times\frac{\sqrt{5}-6i}{\sqrt{5}-6i}=\sqrt{5}-6i$

Hence (a) is the correct option.

Exp. 3) Which of the following is correct?

(a) $7+i>5-i$ (b) $8+i>9-i$

(c) $13+i<25-3i$ (d) none of these

Solution Since the relation is not defined *i.e.*, $(a+ib)\gtreqless(c+id)$ is indeterminable. Hence (d) is the correct option.

Exp. 4) The smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n=-1$ is :

(a) 5 (b) 1

(c) 2 (d) none of these

Solution Given that $\left(\frac{1+i}{1-i}\right)^n=-1$

$\Rightarrow$ $\left[\frac{(1+i)\,(1+i)}{(1-i)\,(1+i)}\right]^n=-1 \Rightarrow i^n=-1$

$\Rightarrow$ $(-1)^{\frac{n}{2}}=(-1) \Rightarrow \frac{n}{2}=1$

$\therefore$ $n=2$

Hence (c) is the correct option.

Exp. 5) The smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n=1$ is:

(a) 2 (b) 4 (c) 6 (d) 8

Solution $\left(\frac{1+i}{1-i}\right)^n=\left(\frac{(i+i)\,(1+i)}{(1-i)\,(1+i)}\right)^n=\left(\frac{(1+i)^2}{2}\right)^n=i^n=1$

$\Rightarrow$ $i^n=1=i^4 \Rightarrow n=4$

Hence (b) is the correct option.

Exp. 6) $(3+2i)^3=?$

(a) $9-46i$ (b) $9+46i$

(c) $-9+46i$ (d) none of these

Solution $(3+2i)^3=3^3+(2i)^3+3.3.2\,i\,(3+2i)$

$=27+8i^3+18i\,(3+2i)=27+8i^3+54i+36i^2$

$=27-8i+54i-36=(-9+46i)$

Hence (c) is the correct option.

Exp. 7) The multiplicative inverse of $(3+2i)^2$ is :

(a) $\frac{12}{169}-\frac{5i}{169}$ (b) $\frac{5}{169}-\frac{12i}{169}$

(c) $\frac{5}{13}-\frac{12i}{13}$ (d) none of these

Solution $Z=(3+2i)^2=9+4i^2+12i=9-4+12i=5+12i$

then $z^{-1}=\frac{1}{5+12i}=\frac{1}{5+12i}\times\frac{5-12i}{5-12i}$

$=\frac{5-12i}{25-144i^2}=\frac{5-12i}{169}=\left(\frac{5}{169}-\frac{12i}{169}\right)$

Thus (b) is the correct option.

Exp. 8) If $|z|=1$ then $\left(\frac{1+z}{1+\bar{z}}\right)$ equals :

(a) $\bar{z}$ (b) z

(c) z^{-1} (d) none of these

Solution $\frac{1+z}{1+\bar{z}}=\left(\frac{1+z}{1+\bar{z}}\right)\frac{z}{z}=\frac{z\,(1+z)}{(z+1)}=z$ $(\because |z|=1)$

Hence (b) is the correct option.

Exp. 9) The value of $\sqrt{i}+\sqrt{-i}$ is :

(a) 1 (b) $\sqrt{2}$

(c) $-i$ (d) none of these

Solution $\sqrt{i}+\sqrt{-i}=\sqrt{(\sqrt{i}+\sqrt{-i})^2}$

$=\sqrt{i-i+2\sqrt{i}\,\sqrt{-i}}=\sqrt{2\sqrt{-i^2}}=\sqrt{2}$

Hence (b) is the correct option.

Exp. 10) The value of $\log_e(-1)$ is :

(a) $i\pi$ (b) 0
(c) i^{-1} (d) does not exist

Solution $\log_e(-1) = \log_e(e^{i\pi}) = i\pi$

Hence (a) is the correct option.

Exp. 11) The number of solutions of the equation $z^2 + |z|^2 = 0$, where z is a complex number is :

(a) 1 (b) 2
(c) 4 (d) infinitely many

Solution $z^2 + |z|^2 = 0 \Rightarrow z^2 + z\bar{z} = 0$

$\Rightarrow$ $z(z + \bar{z}) = 0$

or $z \,.\, 2\text{Re}(z) = 0$

$\therefore$ $z = 0$ and $\text{Re}(z) = 0$

If $z = x + iy \Rightarrow 0 = x + iy \Rightarrow (x, y) = (0, 0)$

and if $z = 0 + ib \Rightarrow (a, b) = (0, b)$ but $b \in R$

So, there are infinitely many solutions. Since $b \in R$ Hence (d) is the correct option.

Exp. 12) The value of $\left(\frac{i+\sqrt{3}}{2}\right)^{100} + \left(\frac{i-\sqrt{3}}{2}\right)^{100}$ is :

(a) -1 (b) 1 (c) 0 (d) i

Solution $\frac{i+\sqrt{3}}{2} = \left(\frac{i+\sqrt{3}}{2}\right)\frac{i}{i} = \left(\frac{-1+\sqrt{3}i}{2i}\right) = \frac{\omega}{i}$

($\because$ A complex number $a + ib$ for which $|a : b| = 1 : \sqrt{3}$ or $\sqrt{3} : 1$ can always be expressed in terms of i, ω or ω^2)

Similarly, $\frac{i-\sqrt{3}}{2} = \frac{\omega^2}{i}$

$\therefore$ $\left(\frac{i+\sqrt{3}}{2}\right)^{100} + \left(\frac{i-\sqrt{3}}{2}\right)^{100} = \left(\frac{\omega}{i}\right)^{100} + \left(\frac{\omega^2}{i}\right)^{100}$

$= \omega^{99}.\omega + \omega^{198}.\omega^2$ $[\because \omega^{99} = \omega^{198} = 1]$

$= \omega + \omega^2 = -1$

Hence (a) is the correct option.

Exp. 13) If $1, \omega, \omega^2$ be the cube roots of unity, then the value of $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5$ is :

(a) 0 (b) 16 (c) 32 (d) 64

Solution $\because$ $1 + \omega + \omega^2 = 0$

$\therefore$ $1 + \omega = -\omega^2$ and $1 + \omega^2 = -\omega$

$\therefore$ $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = (-2\omega)^5 + (-2\omega^2)^5$

$= -32(\omega^5 + \omega^{10})$ $[\because \omega^5 = \omega^2$ and $\omega^{10} = \omega$ and $\omega + \omega^2 = -1]$

$= 32$

Hence (c) is the correct option.

Exp. 14) If $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$ be the n, nth roots of unity, the value of $(9 - \omega)(9 - \omega^2)\ldots(9 - \omega^{n-1})$ will be :

(a) 0 (b) $\frac{9^n - 1}{8}$

(c) $\frac{8^n - 1}{3}$ (d) none of these

Solution Let $x^n = 1 \Rightarrow x = (1)^{1/n}$

$\Rightarrow$ $x^n - 1 = 0$

or $x^n - 1 = (x - 1)(x - \omega)(x - \omega^2)\ldots(x - \omega^{n-1})$

$\Rightarrow$ $\frac{x^n - 1}{x - 1} = (x - \omega)(x - \omega^2)\ldots(x - \omega^{n-1})$

Putting $x = 9$ in both sides, we have

$(9 - \omega)(9 - \omega^2)(9 - \omega^3)\ldots(9 - \omega^{n-1}) = \frac{9^n - 1}{8}$

Hence (b) is the correct choice.

Exp. 15) If $\left(\frac{1-i}{1+i}\right)^{100} = x + iy$ then the value of (x, y) is :

(a) (0, 1) (b) (0, 0)
(c) (1, 0) (d) $(-1, 0)$

Solution $\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{-2i}{2} = -i$

Hence $\left(\frac{1-i}{1+i}\right)^{100} = (-i)^{100} = 1$

Thus $x + iy = 1 \Rightarrow x = 1, y = 0$

So the value of $(x, y) = (1, 0)$

Hence (c) is the correct option.

Exp. 16) The inequality $a + ib < c + id$ holds if

(a) $a < c, b < d$ (b) $a < c, b > d$
(c) $a > c, b < d$ (d) none of these

Solution Option (d) is correct.

Exp. 17) The points z_1, z_2, z_3, z_4 in the complex plane form the vertices of a parallelogram iff :

(a) $z_1 + z_2 = z_3 + z_4$ (b) $z_1 + z_3 = z_2 + z_4$
(c) $z_1 + z_3 > z_2 + z_4$ (d) none of these

Solution If the mid point of z_1z_3 is the same as that of z_2z_4.

i.e., $\frac{1}{2}(z_1 + z_3) = \frac{1}{2}(z_2 + z_4)$

$\Rightarrow z_1 + z_3 = z_2 + z_4$, only then these points can form the parallelogram.

Exp. 18) If $a + ib = \sqrt{\frac{(u+iv)}{(x+iy)}}$ then the value of $a^2 + b^2$ is :

(a) $\sqrt{\frac{u^2+v^2}{x^2+y^2}}$ (b) $\frac{u^2-v^2}{x^2-y^2}$

(c) can't be determined (d) none of these

Solution $a^2 + b^2 = |a + ib|^2$

$= \left|\frac{u+iv}{x+iy}\right|$

(Putting the value of $a + ib$)

$= \frac{\sqrt{u^2+v^2}}{\sqrt{x^2+y^2}}$

Hence (a) is the correct option.

Exp. 19) The maximum value of $|z|$ when z satisfies the condition $\left|z+\frac{2}{z}\right|=2$ is :

(a) $\sqrt{3}-1$ (b) $\sqrt{3}$ (c) $\sqrt{3}+1$ (d) 1

Solution $\because$ $|z|=\left|z+\frac{2}{z}-\frac{2}{z}\right|$

$$|z|\le\left|z+\frac{2}{z}\right|+\left|\frac{2}{z}\right|$$

It means $|z|\le 2+\frac{2}{|z|}$

$\Rightarrow$ $|z|^2\le 2|z|+2$

$\Rightarrow$ $|z|^2-2|z|+1\le 2+1$

$\Rightarrow$ $(|z|-1)^2\le 3$

$\Rightarrow$ $-\sqrt{3}\le|z|-1\le\sqrt{3}$

$\Rightarrow$ $1-\sqrt{3}\le|z|\le 1+\sqrt{3}$

Therefore the maximum value of $|z|$ is $1+\sqrt{3}$

Exp. 20) The modulus of the complex number $\frac{(2-i\sqrt{5})}{(1+2\sqrt{2}i)}$ is :

(a) -1 (b) 1 (c) $\sqrt{3}$ (d) $-\sqrt{3}$

Solution The modulus of $\left(\frac{2-i\sqrt{5}}{1+2\sqrt{2}i}\right)=\left|\frac{2-i\sqrt{5}}{1+2\sqrt{2}i}\right|$

$$=\frac{\sqrt{4+5}}{\sqrt{1+8}}=\frac{3}{3}=1$$

Hence (b) is the correct answer.

Introductory Exercise 1.10

1. Which one of the following is a rational number?
   (a) $(\sqrt{2})^2$ (b) $2\sqrt{2}$ (c) $2+\sqrt{2}$ (d) $\frac{\sqrt{2}}{2}$
2. Rational number $\frac{-18}{5}$ lies between consecutive integers :
   (a) -2 and -3 (b) -3 and -4
   (c) -4 and -5 (d) -5 and -6
3. Which one of the following statements is correct?
   (a) There can be a real number which is both rational and irrational
   (b) The sum of two irrational number is always irrational
   (c) For any real numbers x and y, $x<y\Rightarrow x^2<y^2$
   (d) Every integer is a rational number
4. Which one of the following statements is not correct?
   (a) If 'a' is a rational number and b is irrational, then $a+b$ is irrational
   (b) The product of a non-zero rational number with an irrational number is always irrational
   (c) Addition of any two irrational numbers can be an integer
   (d) Division of any two integers is an integer
5. If x be a rational number and y be an irrational number, then :
   (a) both $x+y$ and xy are necessarily irrational
   (b) both $x+y$ and xy are necessarily rational
   (c) xy is necessarily irrational, but $x+y$ can be either rational or irrational
   (d) $x+y$ necessarily irrational, but xy can be either rational or irrational
6. A rational equivalent to $\frac{-24}{20}$ with denominator 25 is :
   (a) $\frac{-30}{25}$ (b) $\frac{30}{25}$ (c) $\frac{-29}{25}$ (d) $\frac{-19}{25}$
7. Let $\frac{a}{b}=\frac{c}{d}$, (where a and b are odd prime numbers). If $c>a$ and $d>b$, then :
   (a) c is not a multiple of a (b) d is not a multiple of b
   (c) $c=ka$, $d=kb$ with $k>1$
   (d) $c=ka$, $d=lb$ with $k\ne l$

Directions (for Q. Nos. 8 to 10) *Answer the following questions based on the information given below. From the set of first 81 natural numbers two numbers p and q are chosen in order to form the rational number $\frac{p}{q}$ such that p and q are co-prime.*

8. Find the number of values of q so that $\frac{p}{q}$ is always a recurring decimal number.
   (a) 9 (b) 14 (c) 67 (d) 72
9. Find the total number of rational numbers such that $\frac{p}{q}$ is always a terminating decimal number.
   (a) 622 (b) 441 (c) 134 (d) 1134
10. Find the number of non-integral rational numbers $\frac{p}{q}$ between 0 and 1, which can always be expressed as the terminating decimals?
    (a) 167 (b) 144
    (c) 214 (d) 377
11. In the online game of Angry Birds, each bird destroys on an average A number of pigs. If $A=3$. $abbcabbcabbc$ ... and out of a, b and c not more than one digit is 0, which of the following could be the total number of birds?
    (a) 1991 (b) 1998
    (c) 29992 (d) 49995

1.11 Various Number Systems

Decimal Number System

Generally we use decimal system in our day to day mathematical applications *e.g.,* counting of the things, accounting in offices etc. Why we call it decimal system?

Since there we use 10 symbols *viz.,* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 to represent the data, that is why we say that the representation is in decimal system.

The sequence of decimal numbers goes on as 0, 1, 2, 3 4, 5, 6, 7, 8, 9, 10, 11,12, 13, 14, 15, 16, 17, 18, 19, 20, 21.. etc. So after 9, each successive number is a combination of 2 (or more) unique symbols of this system.

The decimal system is a positional-value system in which the value of a digit in a number depends on its position. For example 879. Here 8 represents 800, 7 represents 70 and 9 represents 9 units. In essence, 8 carries the most weight of three digits, it is referred as the most significant digit (MSD). The 9 carries the least weight and is called as the least significant digit (LSD)

i.e., $879 = 8 \times 10^2 + 7 \times 10^1 + 9 \times 10^0$

Similarly $63.78 = 6 \times 10^1 + 3 \times 10^0 + 7 \times 10^{-1} + 8 \times 10^{-2}$

So, this number is actually equal to 6 tens plus 3 units plus 7 tenths plus 8 hundredths.

So $3859276 = 3 \times 10^6 + 8 \times 10^5 + 5 \times 10^4 + 9 \times 10^3 + 2 \times 10^2 + 7 \times 10^1 + 6 \times 10^0$

and $9253.827 = 9 \times 10^3 + 2 \times 10^2 + 5 \times 10^1 + 3 \times 10^0 + 8 \times 10^{-1} + 2 \times 10^{-2} + 7 \times 10^{-3}$

In general, any number is simply the sum of the products of each digit value and its positional (or place) value.

Positional values (weights) →	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
	↓	↓	↓	↓	↓	↓	↓
	9	2	5	3	8	2	7
	↑				↑		↑
	MSD				Decimal point		LSD

Thus a two digit number can be expressed as $x \times 10^1 + y \times 10^0$, where x is the tens and y is the unit digit.

Thus the two digit number xy is expressed as $10x + y$ Similarly a three digit number xyz is expressed as $10^2 x + 10^1 y + 10^0 z$ or xyz is expressed as $100x + 10y + z$ where x, y, z are the hundreds tens and unit digit respectively.

In general an n digit number in decimal representation, can be expressed as

$$k_1 \times 10^{n-1} + k_2 \times 10^{n-2} + k_3 \times 10^{n-3} + \ldots + k_n \times 10^0$$

Important Facts about Decimal Numbers

1. The digits which are always same *i,e,* when they are seen upside down, they appear the same. *e.g.,* 0, 1 and 8.
2. The digits which create confusion when they are written upside down they change their face values. *e.g.,* 6 and 9.
3. When the digits of a two digit number are reversed *i.e.,* when their position is interchanged, then the difference of the original number and this resultant number is always divisible by 9. Also the difference of these two numbers is exactly the product of the difference of the two digits (*i.e.,* difference of their face values) with 9.

   For example ,
   $$\begin{array}{r} xy \\ -\ yx \\ \hline 9\,(x-y) \end{array}$$

   $[\because (10x + y) - (10y + x) = 9(x - y)]$

   e.g., $53 - 35 = 9 \times 2 = 18$ $(\because 5 - 3 = 2)$

   e.g., $82 - 28 = 9 \times 6 = 54$ $(\because 8 - 2 = 6)$

NOTE The sum of these two numbers is always divisible by 11.

4. When the digits of a three digit number are reversed *i.e.,* the unit digit becomes hundreds digit and *vice versa*, then the difference between these two numbers is always divisible by 99. Also the difference of these two numbers *(i.e.,* original and resultant number) is exactly equal to the product of the difference of unit and hundreds digit with 99, *i.e.,* $xyz - zyx = 99(x - z)$

   $[\because (100x + 10y + z) - (100z + 10y + x) = 99(x - z)]$

   For example (i) $852 - 258 = 99 \times 6$ $[\because 8 - 2 = 6]$
   $= 594$

   (ii) $703 - 307 = 99 \times 4$ $[\because 7 - 3 = 4]$
   $= 396$

5. When the digits of a four digit number are reversed *i.e,* unit digit with thousands digit and vice versa and tens digit with hundreds digit and vice versa, then the difference between the original number and the resultant number is always divisible by 9.

Exp. 1) The difference between the highest and lowest two digit numbers is :

(a) 88 (b) 89
(c) 22 (d) 99

Solution Simply $99 - 10 = 89$

Hence (b) is the correct option.

Exp. 2) When a two digit number is reversed, then the new number becomes $\frac{5}{6}$ th of the original number. The original number is :

(a) 56 (b) 45
(c) 48 (d) 54

Solution The best way is to go through options.

Now, we consider those options which are divisible by 6, as, the number is an integer.

Again we pick up only those values which on being reversed (their digits) decreases. So only option (d) is suitable. Check it out.

Alternatively Let the original number be $(10x + y)$ then the new number will be $(10y + x)$, then $(10y + x) = \frac{5}{6}(10x + y)$

$$\Rightarrow \quad 55y = 44x$$

$$\Rightarrow \quad \frac{x}{y} = \frac{5}{4}$$

Thus the only possible values of x, y are 5, 4 respectively.

Hence the original number is 54.

Alternatively When you multiply the given (original) number by $\frac{5}{6}$ then the digit of the resultant value gets interchanged.

Exp. 3) The number of two digit numbers which on being reversed (*i.e.,* their digits exchanged the position) gives out perfect square two digit numbers :

(a) 1 (b) 4
(c) 6 (d) 10

Solution These numbers are 61, 52, 63, 94, 46 and 18. Since these numbers on being reversed give out 16, 25, 36, 49, 64, 81 the two digit perfect square numbers.

Hence (c) is the correct answer.

Exp. 4) The number of two digit numbers which are prime :

(a) 25 (b) 23
(c) 21 (d) can't be determined

Solution There are 25 prime numbers between 1 and 100. There are 4 prime numbers of 1 digit (*viz,* 2, 3, 5, 7)

So, the number of two digit prime numbers $= 25 - 4 = 21$

Hence (c) is the correct option.

Exp. 5) The number of two digit prime numbers, with distinct digits, on being reversed they again give the prime numbers is :

(a) 3 (b) 5
(c) 8 (d) 11

Solution See, the total number of prime numbers in this case (when there are distinct digits *i.e.,* there are no same digits) the number of prime numbers must be an even number.

Hence (c) is the correct option.

Alternatively These numbers are 13, 31, 17, 71, 37, 73 and 79, 97.

Thus there are total 8 such numbers.

Exp. 6) The total number of two digit numbers whose unit digit is either, same, double, triple or quadruple of the tens digit :

(a) 13 (b) 9
(c) 18 (d) 90

Solution 11, 22, 33, 44, 55, 66, 77, 88, 99
12, 24, 36, 48
13, 26, 39
14, 28

Thus there are total 18 such numbers. Hence (c) is the correct answer.

Exp. 7) The number of two digit numbers which are perfect square and perfect cube both is :

(a) 0 (b) 2
(c) can't be determined (d) none of these

Solution Two digit perfect cubes are 27, 64

Two digit perfect squares are 16, 25, 36, 49, 64, 81

Thus there is only one value *i.e.,* 64 which is both perfect square and perfect cube.

Thus (d) is the most appropriate choice.

Exp. 8) Aprajita multiplies a number by 72 instead of 27, then her new answer will increase by, than the actual result :

(a) $\frac{7}{2}$ (b) $\frac{8}{3}$
(c) $\frac{5}{3}$ (d) none of these

Solution Let the number be k then the original product will be $k \times 27$ and the new product will be $k \times 72$.

So, the difference in the product value

$$= 72k - 27k = 45k$$

Thus, new product will increase by $\frac{45k}{27k} = \frac{5}{3}$

Hence (c) is the correct answer.

Alternatively $\frac{72 - 27}{27} = \frac{5}{3}$

Exp. 9) If the numerator and denominator of a fraction are exchanged then the product of these two fractions becomes equal to 1. The total number of such fractions is

(a) 1 (b) 31
(c) 13 (d) infinitely many

Solution Let $\frac{x}{y}$ be the fraction where $x, y \neq 0$, then

$$\frac{x}{y} \times \frac{y}{x} = 1 \quad \text{(always)}$$

Now since x and y can assume infinite values so (d) is the best answer.

Exp. 10) $\frac{3}{4}$th of a number is 20 more than half of the same number. The required number is :

(a) 50 (b) 180 (c) 90 (d) 80

Solution Since the number is an integer so it must be divisible by 4. Hence option (a) and (c) are ruled out.

Now, if we check option (b) we find it is wrong so (d) is the correct.

As $$80 \times \frac{3}{4} = 80 \times \frac{1}{2} + 20 \quad \Rightarrow 60 = 60$$

Alternatively $x \times \frac{3}{4} = x \times \frac{1}{2} + 20 \quad \Rightarrow \quad x = 80$

Exp. 11) When 50 is added to the 50% of a number, then the number becomes itself. The required number is :

(a) 375 (b) 100 (c) 150 (d) 500

Solution $$50 + \frac{x}{2} = x \Rightarrow \frac{x}{2} = 50 \quad \Rightarrow \quad x = 100$$

Hence (b) is the correct option.

Exp. 12) If we reverse the digits of a two digit number then the difference between the original number and new number is 27, the difference between the digits is :

(a) 9 (b) 3
(c) can't be determined (d) none of these

Solution Since $xy - yx = 9(x - y)$

Here $9(x - y) = 27$

Thus $x - y = 3$

So (b) is the correct answer.

Exp. 13) A two digit number is 4 times the sum of its digits and the unit digit is 3 more than the tens digit. The number is :

(a) 52 (b) 61 (c) 63 (d) 36

Solution Go through option $36 = 4(3 + 6)$ and $6 - 3 = 3$

Hence (d) is the correct answer.

Alternatively Let x, y be the tens and unit digits respectively, then

$$(10x + y) = 4(x + y) \quad \ldots\text{(i)}$$

$$\Rightarrow \quad 6x = 3y$$

$$\Rightarrow \quad \frac{x}{y} = \frac{1}{2} \quad \ldots\text{(ii)}$$

and $$y - x = 3 \quad \ldots\text{(iii)}$$

from equation (ii) and (iii)

$$x = 3 \quad \text{and} \quad y = 6$$

Thus the required number is 36.

Exp. 14) When a two digit number is subtracted from the other two digit number which consists of the same digits but in reverse order, then the difference comes out to be a two digit perfect square. The number is :

(a) 59 (b) 73
(c) 36 (d) not unique

Solution Since the difference between number is a perfect square. So this difference can be only 36, because 36 is the only two digit perfect square contains 9 as a factor.

But there are total 5 numbers possible *viz.*, 15, 26, 37, 48, 59. Since the only condition is that

$$(10x + y) - (10y + x) = 9(x - y) = 36$$

$$\Rightarrow \quad (x - y) = 4$$

Thus (d) is the most appropriate answer.

Exp. 15) There is a three digit number such that the sum of its end digits (unit digit and hundredth place digit) is always a single digit number. Another three digit number is obtained by reversing the position of end digits of the original number. Then what can be the possible sum of the tens digits of both these numbers?

(a) 5 (b) 12 (c) 15 (d) 26

Solution Since in both the numbers the middle digits *i.e.*, the tens digits are same, it means the sum of these tens digits is always an even number ($\because x + x = 2x$)

So the option (a) and (c) are ruled out.

Now, since the largest possible digit is 9, so the maximum possible sum of these two digits can be 18 (= 9 + 9). Therefore option (d) is also ruled out.

Thus the possible answer is (b).

Exp. 16) A three digit number which on being subtracted from another three digit number consisting of the same digits in reverse order gives 594. The minimum possible sum of all the three digits of this number is :

(a) 8 (b) 12
(c) 6 (d) can't be determined

Solution Let x, y, z be the hundred, tens and unit digits of the original number then

$$(100z + 10y + x) - (100x + 10y + z) = 594$$

$$\Rightarrow \quad 99(z - x) = 594$$

$$\Rightarrow \quad (z - x) = 6$$

So the possible values of (x, z) are $(1, 7), (2, 8)$ and $(3, 9)$. Again the tens digit can have the values *viz.*, 0, 1, 2, 3, ... 9.

So the minimum possible value of $x + y + z = 1 + 0 + 7 = 8$.

Hence (a) is the correct option.

NOTE x and z can never be zero since if the left most digit becomes zero, then it means this number is only two digit number.

Digital Sum

The digital sum of a number is a single digit number obtained by an iterative (or repeated) process of summing the digits. In this process all the digits are added to form a new number, and then again all the digits of new number are added to form another number. The process continues until a single digit number is obtained.

Exp. 1) Find the digital sum of 7586902.

Solution $7+5+8+6+9+0+2=37$

$3+7=10$

$1+0=1$

Therefore the digital sum of 7586902 is 1.

Additive Persistence The number of times the digits must be summed to reach the digital sum is called a number's additive persistence. In the above example, the additive persistence of 7586902 is 3.

Properties of Digital Roots

1. The digital sum of any number in the decimal system is the same as the remainder obtained when that number is divided by 9. In terms of digital sum the remainder 0 is equivalent to the digit 9.
2. Adding 9 or its multiples to any number does not change the digital sum of that number.
3. Removing the digit 9 or removing the cluster of digits those add up to 9 would not affect the digital sum of the given number.
4. A number is divisible by 3, if its digital sum is 0, 3, 6 or 9.
5. A number is divisible by 9, if its digital sum is 0 or 9.
6. The digital root of a square is 1, 4, 7, or 9. Digital roots of square numbers progress in the sequence 1, 4, 9, 7, 7, 9, 4, 1, 9.
7. The digital root of a perfect cube is 1, 8 or 9, and digital roots of perfect cubes progress in that exact sequence.
8. The digital root of a prime number (except 3) is 1, 2, 4, 5, 7, or 8.
9. The digital root of a power of 2 is 1, 2, 4, 5, 7, or 8. Digital roots of the powers of 2 progress in the sequence 1, 2, 4, 8, 7 and 5. This even applies to negative powers of 2; for example, 2 to the power of 0 is 1; 2 to the power of -1 (minus one) is .5, with a digital root of 5; 2 to the power of -2 is .25, with a digital root of 7; and so on, ad infinitum in both directions. This is because negative powers of 2 share the same digits (after removing leading zeroes) as corresponding positive powers of 5, whose digital roots progress in the sequence 1, 5, 7, 8, 4, 2.
10. The digital root of an even perfect number (except 6) is 1.
11. The digital root of a star number is 1 or 4. Digital roots of star numbers progress in the sequence 1, 4, 1.
12. The digital root of a nonzero multiple of 9 is 9.
13. The digital root of a nonzero multiple of 3 is 3, 6 or 9.
14. The digital root of a triangular number is 1, 3, 6 or 9. Digital roots of triangular numbers progress in the sequence 1, 3, 6, 1, 6, 3, 1, 9, 9.
15. The digital root of a factorial $\geq 6!$ is 9.
16. The digital root of Fibonacci numbers is a repeating pattern of 1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 9.
17. The digital root of Lucas numbers is a repeating pattern of 2, 1, 3, 4, 7, 2, 9, 2, 2,4, 6, 1, 7, 8, 6, 5, 2, 7, 9 7,7, 5, 3, 8.
18. The digital root of the product of twin primes, other than 3 and 5, is 8. The digital root of the product of 3 and 5 (twin primes) is 6.
19. The digital root of a non-zero number is 9 if and only if the number is itself a multiple of 9.

Digital Sum Rule of Multiplication

The digital sum of the product of two numbers is equal to the digital sum of the product of the digital sums of the two numbers.

Exp. 2) The product of 174 and 26 is 4524.

The digital sum of 174 is 3 and that of 26 is 8.

The product of digital sums 3 and 8 is 6.

The digital sum of 4524 is 6.

Therefore the digital sum of 174x26 = Digital sum of 4524.

Determining Whether a Number is a Perfect Square or Not

The digital sum of the numbers that are perfect squares will always be 1, 4, 7, or 9.

However, a number will NOT be a perfect square if its digital sum is NOT 1, 4, 7, or 9, but it may or may not be a perfect square if its digital sum is 1, 4, 7, or 9.

The numbers 1, 81, 1458 and 1729 are each the product of their own digit sum and its reversal, for example $1+7+2+9=19$, and $19 \times 91 = 1729$.

Exp. 3) Find the product of 102030x405060x708090

(a) 39064716832962000
(b) 281635785138962000
(c) 29264135978862000
(d) 291635781359782000